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# LOGISTIC ANALYSIS OF BRUCETON DATA

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Naval Ordnance Laboratory
White Oak, Maryland

23 July 1973

AD 766780

**NOLTR 73-91**

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BY
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23 JULY 1973

NOL

NAVAL ORDNANCE LABORATORY, WHITE OAK, SILVER SPRING, MARYLAND



NOLTR 73-91

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97

UNCLASSIFIED

Security Classification

**DOCUMENT CONTROL DATA - R & D**

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author)<br>Naval Ordnance Laboratory<br>White Oak, Silver Spring, Maryland 20910 | 2a. REPORT SECURITY CLASSIFICATION<br>UNCLASSIFIED<br>2b. GROUP |

3. REPORT TITLE

LOGISTIC ANALYSIS OF BRUCETON DATA

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

L. D. Hampton, G. D. Blum, and J. N. Ayres

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| 23 July 1973 | ~~vi + 90~~ 97 | 8 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO.<br>Task NOL-443/NWL<br>b. PROJECT NO.<br>ORD-033 234/092-1/F008-08-11<br>c.<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>NOLTR 73-91<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY<br>Naval Weapons Laboratory<br>Dahlgren, Virginia 22448<br>Naval Ordnance Systems Command<br>~~Washington, D.C. 20360~~ |

13. ABSTRACT

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DD FORM 1473 (PAGE 1)
1 NOV 65

S/N 0101-807-6801

UNCLASSIFIED

Security Classification

UNCLASSIFIED
Security Classification

| 14 KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Statistics | | | | | | |
| Bruceton | | | | | | |
| Logistic distribution | | | | | | |
| Go/no-go data analysis | | | | | | |
| Monte Carlo | | | | | | |
| Sensitivity | | | | | | |
| Explosives | | | | | | |

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

UNCLASSIFIED
Security Classification

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By

L. D. Hampton, G. D. Blum, J. N. Ayres

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NOLTR 73-91 23 July 1973

LOGISTIC ANALYSIS OF BRUCETON DATA

Previous work has shown that there are numerous occasions where sensitivity data should be interpreted with the logistic rather than the normal (Gaussian) as the assumed distribution function. One of the tools for collecting and assessing sensitivity data is the Bruceton test and analysis method. The analysis, as it was originally developed, assumes the normal distribution.

This report contains a theoretical development and study of a Bruceton analytical method assuming the logistic distribution function. Also, Monte Carlo techniques were used to study the accuracies of the estimating parameters and how the accuracies are affected by sample size. Although the work was begun over seven years ago under the HERO (Hazards of Electromagnetic Radiation to Ordnance) Project, Task NOL-443/NWL and completed somewhat later under ORDTASK ORD-033 234/092-1/F008-08-11 Problem 001, Reliability and Gap Sensitivity of Explosives, it was not published because two of the authors left the Laboratory. The methods described are of continuing utility however and warrant publication.

Since the Bruceton test method is widely used for collecting sensitivity data for predictions of safety and reliability, this work should be of interest to systems analysts as well as to those working directly with explosives in the fields of research, design, and manufacturing quality control.

ROBERT WILLIAMSON II

C. J. ARONSON
By direction

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TABLES

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ILLUSTRATIONS

# I. INTRODUCTION

1. The statistical treatment of experimental results in which the response of an item to a stimulus is of a quantal (yes or no) nature is becoming an increasingly important problem in current research. The Bruceton test was designed to handle just such experiments when the stimulus is continuously controllable[1,2]*. Its sequential, "up-and-down," method of testing is ideally suited to destructive testing (any situation in which each item can only be tested once) because it tends to be economical in the number of items consumed in testing[1]. This economy has led to its being used extensively in explosives technology and, to some degree, in the field of experimental biology/medicine.

2. Another recent development in statistical data analysis is the increased use of the logistic distribution[3]. The expression for the cumulative logistic probability is

$$p(x) = \int_{-\infty}^{x} \frac{1}{2\gamma}\left[\frac{1}{1+\cosh\left(\frac{x-\mu}{\gamma}\right)}\right] dx = \frac{1}{1+\exp-\left(\frac{x-\mu}{\gamma}\right)} \quad . \qquad (1)$$

This distribution has been shown to be preferable to the normal** in certain applications[4,5].

3. Earlier work on the Bruceton test has based the calculations on the assumption that the population from which the items under test are drawn is normally distributed[1,2]. Part II is a recapitulation of the statistical concepts that we use in the ensuing presentations. Those with an active familiarity with the field of statistics of attributes can skim or skip this section. Part III is a discussion of the strategy and technique for carrying out a Bruceton test and analyzing the results. The dissimilarities between the Gaussian-Bruceton and the logistic-Bruceton analyses are pointed out. Two numerical examples are used to demonstrate the [illegible] of a Bruceton analysis. The graphs and tables used in carrying out a logistic Bruceton analysis are collected into one place (Appendix A) along with a summary of appropriate equations to aid in the carrying out of the analyses of experimental data. Part IV of this paper is a redevelopment of the calculational procedure of the Bruceton test using, as a basic assumption, the hypothesis that the population is logistically distributed. Part V is a report of an extensive series of Monte Carlo tests of the logistic-Bruceton analysis, showing the effects of finite sample size and related factors on the theoretical assumptions.

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*References are on page 47.

**The comparable expression for the Gaussian distribution is

$$p(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^{2}\right] dx \qquad (2)$$

## II. STATISTICAL BACKGROUND

### Critical Level

4. Implicit in the following methodology is the concept of "critical level". An explosive system will respond if the appropriate triggering signal is large enough. There is some signal amplitude below which the system will not respond but at which, or above, it will respond. This signal amplitude is the critical level. A group of explosive systems, even when made as nearly identical to each other as humanly possible, cannot be expected to have identical critical levels. In general the critical levels will be grouped around a central value, some being further away (above and below) than others.

### Distribution

5. Experience has shown that this grouping, or distribution, can be represented by a bell-shaped plot, as in Fig 1. Here p(x) represents the relative probability of encountering a device whose critical level is precisely equal to the stimulus, x. If the curve is

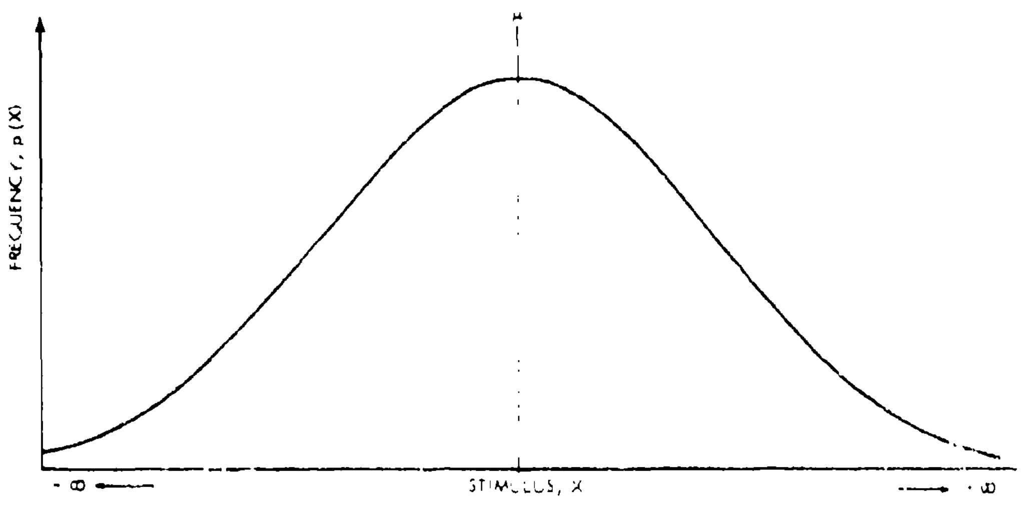


FIG. 1 A PROBABILITY DENSITY FUNCTION (FREQUENCY PLOT)

symmetrical, the median, mode, and mean* all fall on the same stimulus level ($\mu$ in Fig 1). The Greek letter $\mu$ is defined as the mean of the population: the population being all possible devices that have been made, that exist, or that will be made.

Dosage vs Stimulus

6. We distinguish between dosage and stimulus. The term "dosage" is reserved to designate the magnitude of the physical parameter which the experimenter adjusts to apply the desired signal to the explosive system. The stimulus, which is some conformal function of the dosage, serves to transform the probability density function to some assumed distribution, which in the present case is the logistic. Without this transformation the statistical method to be used would not be valid.

7. As an example of this difference: EED's can be fired from a charged capacitor. The dosage might be the potential to which the capacitor is charged, or the stored energy. But from the properties of the logistic p.d.f. (probability density function), namely that the curve approaches zero probability as x approaches the limits of minus and plus infinity, we are confronted by a logical inconsistency. The idea of firing an EED with negative energy has no meaning. By taking the logarithm of the energy as the stimulus we force the lower asymptotic limit to be zero energy. While there are many other transforms which could be used this is one of the simplest and has been used quite widely.

---

*The mean is defined as $(\Sigma x)/n$. That is, if we have a group of values, observations, or measurements ($x_1, x_2, x_3$, etc) the mean would be the total of the magnitudes of the observations divided by the number of observations. The mode is the particular x (or interval about x) for which there is the greatest number of observations. The median is the particular x which divides the group into upper and lower halves. Unless we are dealing with an absolutely symmetrical distribution we cannot expect these three measures of central tendency to be the same (especially for small groups). For instance, in this group of fifteen numbers:

6,6,7,7,7,7,7,8,9,9,9,10,12,15,16

we find that the mean is 9, the mode is 7, and the median is 8. Therefore, the level of stimulus in a Go/No-Go test at which 50% response is observed (or expected) is the median. To refer to this level as the mean is correct only if the distribution is symmetrical about the median.

## Probability Density Function

8. The probability density function is not particularly useful when working with explosives since we can never know the critical level of each individual explosive unit. For instance, let us take a representative sample from the population of Fig 1. To each member of the sample we apply a particular stimulus, $x_1$. Those members whose critical levels are equal to or less than $x_1$ will respond. The number of members responding is represented in Fig 2 by area a. The remaining members of the sample will not respond (area b).

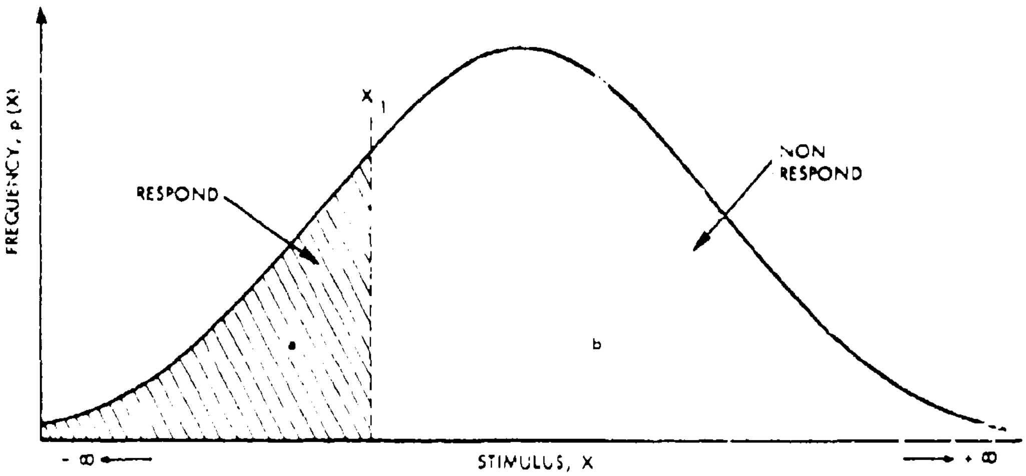


FIG. 2 RESPONSE TO A PARTICULAR STIMULUS

But once the members of area a have responded they no longer exist. We, therefore, cannot find, for any member, whether or not it would have responded at a lower level. Nor can we assume (without some independent source of information) that the members which did not respond were not changed as a result of being subjected to stimulus $x_1$. We can only assume that they may have been changed and therefore are no longer members of the population from which the original sample was drawn.

## Cumulative Probability Function

9. To compute area a, Fig 2, we need to integrate the p.d.f. from the negative limit up to $x_1$. A plot of the integral of the p.d.f. for all values of x ranging from $x = -\infty$ to $x = +\infty$ gives a cumulative curve, such as in Fig 3. From this curve we can deduce the proportion of the population that would respond if the entire population were subjected to a particular stimulus level.* The population proportion that would respond can be considered as either the expected fractional response of a finite sample to a given stimulus or the expected probability of response of a particular item to the same stimulus.

*Deriving the cumulative form by integration of the p.d.f. applies strictly to the Gaussian distribution since, as a matter of history, the logistic distribution was conceived and is only used in the cumulative form.

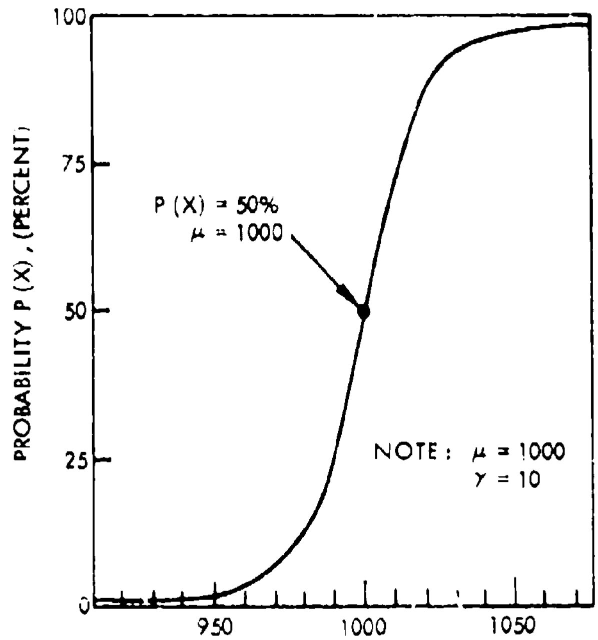


FIG. 3 SIGMOID CUMULATIVE PLOT OF A LOGISTIC DISTRIBUTION

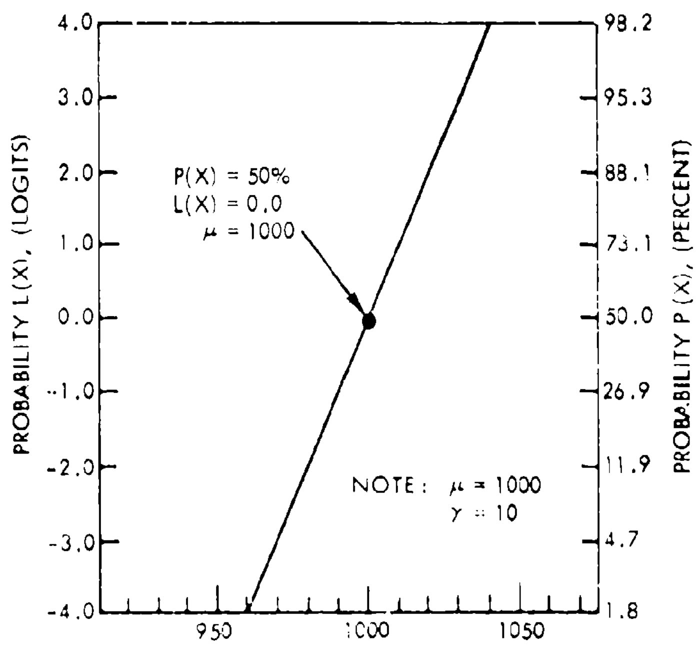


FIG. 4 STRAIGHT LINE CUMULATIVE PLOT OF A LOGISTIC DISTRIBUTION

## The Gaussian and Logistic Distributions

10. In Table 1 we compare the mathematical expressions for the p.d.f.'s and their integrals for the time-honored Gaussian (or normal) distribution and for the logistic function that we will employ in this paper.

Table 1

Comparison of the Gaussian and Logistic Functions

| Gaussian p.d.f. | Logistic p.d.f. |
|---|---|
| $p(x) = \dfrac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]}{\sigma\sqrt{2\pi}}$ (3) | $p(x) = \dfrac{1}{2\gamma}\left(\dfrac{1}{1+\cosh\left(\frac{x-\mu}{\gamma}\right)}\right)$ (4) |
| **Gaussian Cumulative Form** | **Logistic Cumulative Form** |
| $p(x) = \displaystyle\int_{-\infty}^{x} \dfrac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dx}{\sigma\sqrt{2\pi}}$ (2) | $p(x) = \dfrac{1}{1+\exp\left[-\left(\frac{x-\mu}{\gamma}\right)\right]}$ (1) |

The evaluation of the Gaussian integral, in tabular form, is available in many collections of mathematical tables. Because the integral cannot be evaluated in closed form the use of the Gaussian function in high-speed computer programs depends upon either tabular look-up (which is apt to be inefficient in computer storage space and processing time) or else approximation expressions (which have to be devised or found). On the other hand, the logistic cumulative and its transformations involve only logarithmic and exponential functions which are already available in most machine languages.

## Probability Space

11. Even the cumulative curve shown in Fig 3 is not in its most useful form. The vertical axis can be transformed so that the "ess"-shaped curve changes to a straight line, as in Fig 4. We shall use the symbol, L(x), to indicate the logit of x ··· the value of the transformation of p(x). The logit transform can be expressed in a number of equivalent forms:

$$L(x) = \ln\left(\frac{p(x)}{1-p(x)}\right) \tag{5a}$$

$$= \ln \frac{p(x)}{q(x)} \tag{5b}$$

$$= \frac{x-\mu}{\gamma} \tag{5c}$$

where $\mu$ is the mean and $\gamma$ is the variability parameter. By simple algebraic manipulations the preceding equations can be solved for $p(x)$:

$$\frac{p(x)}{1-p(x)} = \exp\left(\frac{x-\mu}{\gamma}\right) \tag{6}$$

$$p(x) = \frac{1}{1+\exp\left[-\left(\frac{x-\mu}{\gamma}\right)\right]} \tag{7}$$

which can be differentiated, as is shown in Appendix B, to yield the probability density function, Eq (4) or Eq (B-13). Even though Berkson[8] expressed the logit transform as a conventional linear equation, $L(x) = Ax+B$, we prefer and will use the form of Eq's (5). The nature of the straight-line transformation can be better understood by making substitutions in the equations. If we let $p(x) = 0.5$ then the logit of x will be the natural logarithm of 1, or zero. If we let x be larger than $\mu$ by one gamma-unit (this is comparable to "one sigma above the mean" in the normal distribution) we would find that

$$L(x) = \frac{(\mu+\gamma)-\mu}{\gamma} = 1.$$

As a consequence, $p(x)/q(x) = \exp(1) = 2.71828$. Hence, $p(x)$ must be 0.73106.

12. A similar y-axis transformation can be made to produce a straight line for the Gaussian cumulative function. The units of the vertical coordinate usually have been designated normits, with zero normits being made to correspond to a probability of 0.50. Finney, in his probit analysis, uses units of probits which are numerically 5.0 greater than the normit.

## Slope and Variance

13. The slope of the normal cumulative curve (in the normal probability space) is $1/\sigma$ where $\sigma^2$ is the variance. The slope of the logistic cumulative curve (in the logistic probability space) is $1/\gamma$. The expression for the variance of the logistic distribution is given by the following equation

$$\sigma^2 = \frac{\pi^2 \gamma^2}{3} .$$

(The derivation of this expression, as well as other moments of the p.d.f., is given in Appendix B.) Table 2 has been included to show the relationships between the logistic and two Gaussian straight line coordinate systems and their corresponding probabilities.

Table 2

Various Y-Axis Parameters for the Gaussian and Logistic Probability Spaces

| Gaussian Probability | | | Logistic Probability | |
|---|---|---|---|---|
| N Normit | P Probit | Cumulative Probability | L Logit | Cumulative Probability |
| 4 | 9 | 0.999683 | 4 | 0.982014 |
| 3 | 8 | 0.998650 | 3 | 0.952574 |
| 2 | 7 | 0.977250 | 2 | 0.880797 |
| 1 | 6 | 0.84134 | 1 | 0.731059 |
| 0 | 5 | 0.500000 | 0 | 0.500000 |
| -1 | 4 | 0.15866 | -1 | 0.268941 |
| -2 | 3 | 0.022750 | -2 | 0.119203 |
| -3 | 2 | 0.001350 | -3 | 0.047426 |
| -4 | 1 | 0.000317 | -4 | 0.017986 |

## III. BRUCETON METHODS FOR DATA COLLECTION AND ANALYSIS

14. The Bruceton plans for collecting and analyzing data are described in the literature[1,2]. However, at least some of these sources may not be readily available. We have decided to include an exposition of the procedures so that this report can be self-sufficient. Some of the notation has been changed to exploit the advantages of the logistic distribution.

### Strategy

15. The goal of the test method is to concentrate the trials at a few levels centered about $\bar{x}$, the stimulus which will cause 50% response. The units are tested one at a time, the test level being selected in a manner which is most apt to cause a response opposite to the one just observed. The Bruceton data collection plan is carried out in the following way:

   a. Select a number of equally spaced stimulus levels $x_1$, $x_2$, $x_3$, ....... $x_n$ where the lowest and highest are about equidistant from the expected $\bar{x}$ (50% response level.) Let d represent the step size (the difference in magnitude of stimulus between any two adjacent levels).

   b. Choose a level (closest to the expected 50% point) and test the first item at this level.

   c. Test all succeeding items, one at a time, by repeated applications of the following process:

      (1) Note the level at which the last item was tested and note the behavior of this item;

      (2) If the item responded, test the next one at the next lower level; or

      (3) If the item did not respond, test the next one at the next higher level.

      (4) Record the result as a response or non-response at the appropriate level.

16. Recalling that stimulus is defined in such a way that we expect a greater probability of response with an increased (higher level of) stimulus we see that the above procedure concentrates the testing around the 50% response point. If, for instance, we happen to be far above $\bar{x}$, the probability of response will be quite close to 1.0 which means that (by application of rule c.(2) the succeeding test level will probably be closer to $\bar{x}$. Similarly, a trial far below $\bar{x}$ would be expected to cause the succeeding level to be higher than the preceding test level. Use of the Bruceton method causes a hovering around $\bar{x}$ leading to a zig-zag pattern when the results are recorded sequentially on a tally sheet such as in Fig 5.

| | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| STIMULUS | 4.00 | X | | | | | | | | | | | | | | | | | | | | | |
| | 3.80 | | X | | | | | | | | | | | | | | | | | | | | |
| | 3.60 | | | X | | | | X | | | | | | | | X | | | | X | | | |
| | 3.40 | | | | X | | O | | X | | X | | X | | O | | X | | O | | X | | X |
| | 3.20 | | | | | O | | | | O | | O | | O | | | | O | | | | O | |
| | 3.00 | | | | | | | | | | | | | | | | | | | | | | |
| | TRIAL NO. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |

| | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| STIMULUS | 4.00 | | | | | | | X | | | | | | | | | | | | | | | |
| | 3.80 | | | | | | O | | X | | X | | | | | | | | | | | | |
| | 3.60 | | | | | O | | | | O | | X | | X | | X | | X | | X | | X | |
| | 3.40 | | | | O | | | | | | | | O | | O | | O | | O | | O | | O |
| | 3.20 | X | | O | | | | | | | | | | | | | | | | | | | |
| | 3.00 | | O | | | | | | | | | | | | | | | | | | | | |
| | TRIAL NO. | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |

NOTE: X SIGNIFIES A RESPONSE
O SIGNIFIES A NON-RESPONSE

FIG. 5 TALLY OF TYPICAL RESULTS OF A BRUCETON TEST

## Initial Conditions for Bruceton Test

17. Three choices must be made before a Bruceton test can be performed:

a. the proper dosage-stimulus transform (paragraph 6);

b. the starting level; and

c. the step size.

18. In order to use the Bruceton method for analyzing the data, the test levels must be set at equal spacings of increasing intensity of stimulus. Under actual testing conditions we sometimes find (particularly when the step size is small compared to the mean) that there is no practical difference between linear and logarithmic steps. In such a case no transform is necessary. In some cases logarithmic steps would complicate the experiment, as for instance, adjusting the drop height on a stab primer test machine which has a ratchet-detent mechanism with stops at quarter-inch intervals. In such cases, it should be possible to choose a set of logarithmically spaced numerical values which will have a minimum deviation from the experimental levels. This will introduce an error which will have to be lived with.

19. For best efficiency the starting level should be close to $\tilde{x}$. Data obtained previous to the first alternation of responses and non-responses should not be used in the data analysis. For example, in Fig 5 trials 4 and 5 are the first alternation (reversal), and, as a consequence, trials 1, 2, and 3 must be discarded. The early encounter of the first reversal is made more likely by starting the test close to the 50% point. Also, as has been brought out in a previous study[*], a starting level some distance away from the mean tends to bias the estimate of the mean for small-sample Brucetons.

20. The choice of d, the step size, depends upon what kind of information is needed and often entails a compromise. If a close estimate of the mean is desired without much accuracy in the estimate of the standard deviation, then d might be 1/4 to 1/2 of $\sigma$ (if a Gaussian distribution is assumed) or 1/2 to 1 of $\gamma$ (for a logistic distribution). A better estimate of the standard deviation with a corresponding **sacrifice** of an accurate estimate of the mean would require the use of d about equal to $2\sigma$ (or $4\gamma$). When one desires estimates of both parameters then the best compromise seems to be to set d about equal to $\sigma$ (or $2\gamma$). As a passing note we caution that a single run of six or more steps with obviously different 50% points, before and after the run, may indicate that the experiment is out of control.

21. If the step size is very large compared to the variability parameter, an alternating two-level pattern may be obtained. This will happen when one level is far below $\tilde{x}$ and the next higher level is far above $\tilde{x}$. The probability of observing any responses at the lower level or any non-responses at the higher level is very small.

On the other hand, a three-level pattern may evolve when the middle level is close enough to $\bar{x}$ that a mixed response can be expected, and the other levels are so far removed from $\bar{x}$ that the probability of observing all responses on one and all non-responses on the other is very large.

22. In either of these cases (where the step size is large) we can deduce that we have found the limits within which $\bar{x}$ is located. We also may be able to guess at an upper limit for the size of the variability parameter. But these estimates and guesses may not be very informative unless the size of the step is small enough, physically, compared to the desired accuracy.

23. Confidence limits can be assigned to a stimulus, $x_p$, associated with an expected response of p as follows. (A numerical example is given in steps 9 and 10 of the first illustrative example, paragraph 26.) The quantities $s_m$ and $s_g$ are calculated in the first part of the analysis and from these we derive $s_p$ using the variance* equation:

$$s_p^2 = s_m^2 + \left[L(x_p)\right]^2 s_g^2 \quad . \tag{9}$$

Having obtained $s_p$ we can then write the lower and upper confidence limits for $s_p$ as

$$L = x_p - ts_p \tag{10a}$$

$$U = x_p + ts_p \tag{10b}$$

where the t is the Student's t with the assumed confidence and with the proper number of degrees of freedom. For 95% two-sided confidence limits we should enter a t table with P = 97.5% since this gives us 2.5% outside each side of the stated limits. The authors of Reference 1 assumed a very large sample, therefore they used t with an infinite number of degrees of freedom. For small samples which are actually used in experimental work it would be better to use the value of t with the number of degrees of freedom equal to the number of fires or fails observed (whichever was used in the computations). For high-speed computer use we can approximate the values of t quite well by the algorithm

$$t = R + \frac{1}{Sf-V} \tag{11}$$

*$s_g$, $s_m$, and $s_p$ are the variabilities of the estimate of Y, of the mean, and of any particular per cent response point respectively. The variance equation referred to can be found on page 6 of Ref. 6.

where f is the number of degrees of freedom and the values of R, S, and V depend upon the desired value of P. The following are examples:

| P | R | S | V |
|---|---|---|---|
| 90 | 1.282 | 1.2228 | 1.1345 |
| 95 | 1.645 | 0.6602 | 0.6279 |
| 97.5 | 1.960 | 0.4210 | 0.4774 |

The resulting values agree in general with the tabulated values of t when $f \geq 6$ for P = 97.5; for $f \geq 7$ for P = 95; and for $f \geq 9$ for P = 90. Table 3 gives t values for five values of P and compares the results of using the algorithm with published tabular values. Note that the constant R is actually $t_\infty$. We emphasize that the tabular values are the correct values and that this algorithm is an approximation designed for computer use.

## The Bruceton Analysis Method

24. The objective of any Bruceton experiment is to obtain $\bar{x}$ and g (the estimates of the population parameters $\mu$ and $\gamma$) and statistical parameters which can be used to give the probable error of these estimates. Note that the Greek letters $\mu$ and $\gamma$ are reserved for the population parameters whose values are, in real life, not available to the experimenter. The computational procedure is quite simple, starting with a tabulation of the usable fires and fails observed* at the various test levels. As an example, the data of Fig 5 would be tabulated as follows:

| Stimulus x | Number of Responses $n_{x,i}$ | Number of Non-Responses $n_{o,i}$ |
|---|---|---|
| 4.00 | 1 | 0 |
| 3.80 | 2 | 1 |
| 3.60 | 9 | 2 |
| 3.40 | 7 | 10 |
| 3.20 | 1 | 7 |
| 3.00 | 0 | 1 |
| | $\Sigma n_{x,i} = 20$ | $\Sigma n_{o,i} = 21$ |

25. For the analysis a choice is made between responses or non-responses, whichever has the lesser total quantity. In the example the responses therefore are chosen. Each test level is assigned an index

*In this report the number of fires at a particular level (the i-th level) will be represented by a double subscript: $n_{x,i}$; and the number of fails, similarly: $n_{o,i}$.

TABLE 3

ALGORITHM FOR COMPUTATION OF t WITH f DEGREES OF FREEDOM FOR SELECTED VALUES OF P

Equation: $t=R+1/(Sf-V)$

| | P=90% | | P=95% | | P=97.5% | | P=99% | | P=99.5% | |
|---|---|---|---|---|---|---|---|---|---|---|
| R | 1.282 | | 1.645 | | 1.960 | | 2.326 | | 2.576 | |
| S | 1.2228 | | 0.6602 | | 0.4210 | | 0.26687 | | 0.20276 | |
| V | 1.1345 | | 0.6279 | | 0.4774 | | 0.37980 | | 0.33618 | |
| f | Calc | Tab | Calc | Tab | Calc | Tab | Calc | Tab | Calc | Tab |
| 5 | 1.4828 | 1.476 | 2.0190 | 2.015 | 2.5744 | 2.571 | 3.3736 | 3.365 | 4.0518 | 4.032 |
| 6 | 1.4432 | 1.440 | 1.9450 | 1.943 | 2.4481 | 2.447 | 3.1447 | 3.143 | 3.7119 | 3.707 |
| 7 | 1.4166 | 1.415 | 1.8954 | 1.895 | 2.364[illegible] | 2.365 | 2.[illegible]979 | 2.998 | 3.4992 | 3.499 |
| 8 | 1.3976 | 1.397 | 1.8598 | 1.860 | 2.3059 | 2.306 | 2.8958 | 2.896 | 3.3537 | 3.355 |
| 9 | 1.3833 | 1.383 | 1.8331 | 1.833 | 2.2619 | 2.262 | 2.[illegible]206 | 2.821 | 3.2477 | 3.250 |
| 10 | 1.3721 | 1.372 | 1.8123 | 1.812 | 2.2279 | 2.228 | 2.7629 | 2.764 | 3.1672 | 3.169 |
| 15 | 1.3401 | 1.341 | 1.7528 | 1.753 | 2.1313 | 2.131 | 2.6020 | 2.602 | 2.9457 | 2.947 |
| 20 | 1.3248 | 1.325 | 1.7245 | 1.725 | 2.0859 | 2.086 | 2.5277 | 2.528 | 2.8449 | 2.845 |
| 25 | 1.3159 | 1.316 | 1.7079 | 1.708 | 2.0595 | 2.060 | 2.4849 | 2.485 | 2.7873 | 2.787 |
| 30 | 1.3101 | 1.310 | 1.6971 | 1.697 | 2.0422 | 2.042 | 2.4571 | 2.457 | 2.7500 | 2.750 |
| 40 | 1.3029 | 1.303 | 1.6837 | 1.684 | 2.0211 | 2.021 | 2.4231 | 2.423 | 2.7046 | 2.704 |
| 50 | 1.2986 | 1.298 | 1.6758 | 1.676 | 2.0086 | 2.009 | 2.4031 | 2.403 | 2.6780 | 2.678 |
| 60 | 1.2958 | 1.296 | 1.6706 | 1.671 | 2.0003 | 2.000 | 2.3900 | 2.390 | [illegible].6605 | 2.660 |
| 80 | 1.2923 | 1.292 | 1.6641 | 1.664 | 1.9901 | 1.990 | 2.3737 | 2.374 | 2.6390 | 2.639 |
| 100 | 1.2902 | 1.290 | 1.6602 | 1.660 | 1.9840 | 1.984 | 2.3640 | 2.365 | 2.6262 | 2.626 |
| 200 | 1.2861 | 1.286 | 1.6526 | 1.653 | 1.9719 | 1.972 | 2.3449 | 2.345 | 2.6009 | 2.601 |
| 500 | 1.2836 | 1.283 | 1.6480 | 1.648 | 1.9647 | 1.965 | 2.3335 | 2.334 | 2.5859 | 2.586 |
| ∞ | 1.2820 | 1.282 | 1.6450 | 1.645 | 1.9600 | 1.960 | 2.3260 | 2.326 | 2.5760 | 2.576 |

Note: This is a single-sided t table.

number, i, which increases in integral steps per level with increasing intensity of stimulus. Customarily the zero-th level ($i = 0$) is assigned to the lowest level at which tests were carried out, but it can be put wherever the experimenter wishes. By setting $i = 0$ at the level at (or near) which the largest number of observations was made, the sizes of the numbers are reduced, often making the computations possible by inspection or simple mental arithmetic. The two methods will be carried out side by side to facilitate comparison. The derivations of the equations and detailed explanations of such variables as M, D, E, $\delta$, G, and H will be given in Part IV of this report.

## Illustrative Computations, First Example

26. Step 1. Assignment of indices, $x_0$ and d.

$$d = x_1 - x_0 \tag{12}$$

$$\therefore d = 0.20$$

| x | i | $n_{x,i}$ |
|---|---|---|
| 4.00 | 5 | 1 |
| 3.80 | 4 | 2 |
| 3.60 | 3 | 9 |
| 3.40 | 2 | 7 |
| 3.20 | 1 | 1 |
| $x_0 =$ 3.00 | 0 | 0 |

| x | i | $n_{x,i}$ |
|---|---|---|
| 4.00 | 2 | 1 |
| 3.80 | 1 | 2 |
| $x_0 =$ 3.60 | 0 | 9 |
| 3.40 | -1 | 7 |
| 3.20 | -2 | 1 |
| 3.00 | -3 | 0 |

Step 2. Computations of N, A, and B.

| i | $n_{x,i}$ | $in_{x,i}$ | $i^2 n_{x,i}$ |
|---|---|---|---|
| 5 | 1 | 5 | 25 |
| 4 | 2 | 8 | 32 |
| 3 | 9 | 27 | 81 |
| 2 | 7 | 14 | 28 |
| 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 |
| | N=20 | A=55 | B=167 |

| i | $n_{x,i}$ | $in_{x,i}$ | $i^2 n_{x,i}$ |
|---|---|---|---|
| 2 | 1 | 2 | 4 |
| 1 | 2 | 2 | 2 |
| 0 | 9 | 0 | 0 |
| -1 | 7 | -7 | 7 |
| -2 | 1 | -2 | 4 |
| | N=20 | A= -5 | B=17 |

(26. Illustrative Computations, First Example, Continued.)

Step 3. Computation of $\bar{x}$, the stimulus for 50% response.

If $N = \Sigma n_{x,i}$ (for the fires) then $\bar{x} = x_0 + d\left(\frac{A}{N} - \frac{1}{2}\right)$. (13a)

If $N = \Sigma n_{o,i}$ (for the fails) then $\bar{x} = x_0 + d\left(\frac{A}{N} + \frac{1}{2}\right)$. (13b)

using the fires:

$$\bar{x} = x_0 + d\left(\frac{A}{N} - \frac{1}{2}\right) \qquad \bar{x} = x_0 + d\left(\frac{A}{N} - \frac{1}{2}\right)$$

$$\bar{x} = 3.00 + 0.20\left(\frac{55}{20} - \frac{1}{2}\right) \qquad \bar{x} = 3.60 + 0.20\left(-\frac{5}{20} - \frac{1}{2}\right)$$

$$\bar{x} = 3.45 \qquad \bar{x} = 3.45$$

Note that $\bar{x}$ is the same for both sets of indices.

Step 4. Computation of M.

$$M = \frac{B}{N} - \left(\frac{A}{N}\right)^2 \qquad M = \frac{B}{N} - \left(\frac{A}{N}\right)^2 \qquad (14)$$

$$M = \frac{167}{20} - \left(\frac{55}{20}\right)^2 \qquad M = \frac{17}{20} - \left(\frac{5}{20}\right)^2$$

$$M = 8.35 - 7.5625 \qquad M = 0.85 - 0.0625$$

$$M = 0.7875 \qquad M = 0.7875$$

Note again that the difference in indices did not change the final answer.

(26. Illustrative Computations, First Example, Continued.)

Step 5. Determination of D.

$$D = \frac{|x' - \bar{x}|}{d} \quad , \tag{15}$$

where x' is the test level closest to $\bar{x}$.

$$D = \frac{|3.40 - 3.45|}{0.20} = 0.25 \quad .$$

The value of D will have the limits $0.0 \leq D \leq 0.5$ and will not be needed if M is greater than 0.65, which is the case in the present example.

Step 6. Determination of E.

$$E = M, \text{ for the limits of } 0.65 < M < 10, \tag{16a}$$

$$E = M + \delta, \text{ when } M \leq 0.65 \; . \tag{16b}$$

The correction term $\delta$ is a function of both M and D and can be found either from Fig A-2 or in Table A-1. For M greater than 10 the value of E begins again to depend upon both M and D. However, when M approaches 10 it is known that a much too small step size is being used. Since the results of such a test are apt to be very misleading we have stopped the tables at this point.

(26. Illustrative Computations, First Example, Continued.)

Step 7. Computation of $g$, $s_m$, and $s_g$.

$$g = E \cdot d \quad , \tag{17}$$

$$s_m = (G \cdot g)/\sqrt{N}, \text{ and} \tag{18}$$

$$s_g = (H \cdot g)/\sqrt{N} \quad , \tag{19}$$

where G and H are functions of E alone, when $E > 0.65$. When $E \leq 0.65$, G and H are functions of both E and D. The values of G and H are given in Tables A-2 and A-3 and plotted in Figures A-3 and A-4 in Appendix A.

For the numerical example:

$$g = (0.7875)(0.2) = 0.1575$$
$$s_m = (1.630)(0.1575)/(4.472) = 0.05741$$
$$s_g = (1.626)(0.1575)/(4.472) = 0.05726 \ .$$

Step 8. Computation of stimulus, $x_p$, at a specified probability, p.

Eq. (5), $L(x) = \ln\left(\frac{p(x)}{q(x)}\right) = \frac{x-\mu}{\gamma}$ can be rewritten in terms of p and the estimates of the population parameters and solved for $x_p$:

$$\ln\left(\frac{p}{1-p}\right) = \frac{x_p - \bar{x}}{g} \tag{20}$$

$$x_p = \bar{x} + g \cdot \ln\left(\frac{p}{1-p}\right) = \bar{x} + g \cdot L(x) \quad . \tag{21}$$

For example, suppose we wish to estimate the 99% point:

$$x_p = 3.45 + 0.1575\left(\ln \frac{99}{1}\right)$$
$$= 3.45 + 0.1575(4.5951)$$
$$= 4.174 \quad .$$

Note that $L(x_p) = 4.5951$.

(26. Illustrative Computations, First Example, Continued.)

Step 9. Computation of $s_p$.

The variance associated with the estimated stimulus, $x_p$, is given by applying Eq. (9):

$$s_p^2 = s_m^2 + \left[L(x_p)\right]^2 s_g^2 \quad .$$

The term, $L(x_p)$, has been evaluated in Step 8. The standard deviations of the mean and of g have been determined in Step 7. (Note that when $p = 0.50$, $L(x_p) = 0$ and $s_{0.50}$ properly reduces to $s_m$.) Following along with the illustration we find that

$$s_p = \sqrt{(0.05741)^2 + (4.5951)^2(0.05726)^2} = 0.2693.$$

Step 10. Two-sided confidence interval about $x_p$.

If, for instance, we wish to find the two-sided 95% confidence interval about the 99% point we would use

$$(L,U)_{95} = (x_p - t_{97.5}\, s_p,\ x_p + t_{97.5}\, s_p) \qquad (10)$$

where $x_p$ is the 99% point and t is the 97.5% Student's value, with f degrees of freedom where f=N=20. The value of t (from Table 3) is 2.0859; and $x_p$, from Step 8 above, is 4.174. Hence,

$$L_{95} = 4.174 - (2.086)(0.2693) = 3.61$$

$$U_{95} = 4.174 + (2.086)(0.2693) = 4.74 \quad .$$

Step 11. Single-sided limit on $x_p$.

If, instead of an error band or tolerance interval about $x_p$, we wish to find only an upper limit (or only a lower limit) for the estimate of a particular response level we would use either

$$(U)_{95} = x_p + t_{95}s_p, \text{ or} \tag{22a}$$

$$(L)_{95} = x_p - t_{95}s_p \quad . \tag{22b}$$

Thus for the example we would use a value of t (with 20 degrees of freedom since N = 20) of 1.725 which would give

$$(U)_{95} = 4.174 + (1.725)(0.2693)$$

$$= 4.64$$

Notice that the single-sided upper 95% limit — 4.64 — is closer to the value of $x_p$— 4.174 — than the upper two-sided 95% limit — 4.74. This is as it should be.

## Illustrative Computations, Second Example

27. A second example is included to show the procedure when M happens to come out less than 0.65. The data are shown in Fig 6.

Step 1. Assignment of indices, $x_0$ and d.

| | x | i | $n_{o,i}$ |
|---|---|---|---|
| | 3.80 | 2 | 0 |
| | 3.60 | 1 | 2 |
| $x_0$ = | 3.40 | 0 | 13 |
| | 3.20 | -1 | 1 |

Note: Use "fails" since there are fewer "fails" than "fires". $x_0$ = 3.40; d = 0.20.

Step 2. Computation of N, A, and B.

| i | $n_{o,i}$ | $in_{o,i}$ | $i^2n_{o,i}$ |
|---|---|---|---|
| 2 | 0 | 0 | 0 |
| 1 | 2 | 2 | 2 |
| 0 | 13 | 0 | 0 |
| -1 | 1 | -1 | +1 |
| | N=16 | A=1 | B=3 |

| STIMULUS | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3.80 | | | | | | | | X | | | | | | | | | |
| | 3.60 | X | | X | | X | | O | | X | | X | | X | | | | X |
| | 3.40 | | O | | O | | O | | | | O | | O | | X | | O | |
| | 3.20 | | | | | | | | | | | | | | | O | | |
| | TRIAL NO. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |

| STIMULUS | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3.80 | | | | | | | | | | | X | | | | | | |
| | 3.60 | | X | | X | | X | | X | | O | | X | | X | | X | |
| | 3.40 | O | | O | | O | | O | | O | | | | O | | O | | |
| | 3.20 | | | | | | | | | | | | | | | | | |
| | TRIAL NO. | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | |

FIG. 6 TALLY SHEET FOR SECOND BRUCETON EXAMPLE

(27. Illustrative Computations, Second Example, Continued.)

Step 3. Computation of $\bar{x}$.

$$\bar{x} = x_0 + d\left(\frac{A}{N} + \frac{1}{2}\right) \tag{13b}$$

$$= 3.40 + 0.2\left(\frac{1}{16} + \frac{1}{2}\right)$$

$$= 3.5125.$$

Step 4. Computation of M.

$$M = \frac{B}{N} - \left(\frac{A}{N}\right)^2 = \frac{BN-A^2}{N^2} \tag{14}$$

$$= \frac{(3)(16)-1}{(16)(16)}$$

$$= 0.1836.$$

Step 5. Determination of D.

$$D = \frac{|x'-\bar{x}|}{d}$$

$$= \frac{|3.60-3.5125|}{0.2}$$

$$= 0.4375.$$

Step 6. Determination of E

$$E = M + \delta$$

From the graph, Fig A-2, or by double interpolation from Table A-1, we find that $\delta$=0.0409 and E=0.225.

(27. Illustrative Computations, Second Example, Continued.)

Tabular $\delta$ Values

| D \ M | 0.1800 | 0.1850 |
|---|---|---|
| 0.4000 | 0.0399 | 0.0384 |
| 0.5000 | 0.0456 | 0.0440 |

We show the actual process of interpolation:

| D \ M | 0.1800 | 0.1836 | 0.1850 |
|---|---|---|---|
| 0.4000 | 0.0580 | 0.0388* | 0.0384 |
| 0.4325 | | 0.0409** | |
| 0.5000 | 0.0456 | 0.0444* | 0.0440 |

*First Interpolation
**Second Interpolation

Step 7. Computation of $g$, $s_m$, $s_g$.

$$g = E \cdot d \tag{17}$$
$$= (0.225)\ (0.2)$$
$$= 0.0450$$

From the graph, Fig A-3, or by double interpolation from Table A-2, we find that G = 2.4930.

(27. Illustrative Computations, Second Example, Continued.)

Tabular G Values

| D \ E | 0.2000 | 0.2500 |
|---|---|---|
| 0.4000 | 2.666 | 2.265 |
| 0.5000 | 2.769 | 2.306 |

$$s_m = \frac{G \cdot g}{\sqrt{N}}$$

$$s_m = \frac{(2.4930)(0.0450)}{\sqrt{16}} \qquad (18)$$

$$= 0.028$$

Similarly from Figure A-4 or Table A-3, H = 1.146:

Tabular H Values

| D \ E | 0.2000 | 0.2500 |
|---|---|---|
| 0.4000 | 1.141 | 1.174 |
| 0.5000 | 1.105 | 1.140 |

$$s_g = \frac{H \cdot g}{\sqrt{N}} \qquad (19)$$

$$= \frac{(1.1458)(0.0449)}{\sqrt{16}}$$

$$= 0.0129 \quad .$$

(27. Illustrative Computations, Second Example, Continued.)

Step 8. Computation of $x_p$.

Take p, let us say, as 0.95 (the 95% point):

$$x_p = \bar{x} + g \cdot \ln\left(\frac{p}{1-p}\right) \tag{21}$$

$$= 3.5125 + 0.0449\ (\ln 19)$$

$$= 3.6447\ .$$

Step 9. Computation of $s_p$.

$$s_p = \sqrt{(s_m)^2 + \left[\ln\left(\frac{p}{1-p}\right)\right]^2 (s_g)^2}$$

$$= \sqrt{(0.0280)^2 + (2.944)^2 (0.0129)^2}$$

$$= 0.0472\ .$$

(27. Illustrative Computations, Second Example, Continued.)

Ste 10. Two-sided confidence interval about the 95% point. Take the confidence level to be 90%. The student's t value will be at a 95% level for $f = N = 16$

$$\therefore\ t = 1.645 + \frac{1}{(0.6602)(16) - 0.6279} \qquad (11)$$

$$= 1.7457$$

$$L_{90} = 3.5125 - (1.7457)(0.0472)$$

$$= 3.43$$

$$U_{90} = 3.5125 + (1.7457)(0.0472)$$

$$= 3.60$$

Step 11. Single-sided lower limit on the 95% point at 90% confidence

We find that:

$$t = 1.282 + \frac{1}{(1.2228)(16) - 1.1345}$$

$$= 1.3363$$

The lower limit, then is:

$$L_{90} = 3.5125 - (1.3363)(0.0472)$$

$$= 3.45$$

## IV. THEORETICAL STUDIES FOR THE LARGE SAMPLE LOGIT BRUCETON

28. In Section III procedures, tables, and graphs were given to estimate the statistical parameters $\bar{X}$, g, $s_m$, and $s_g$ from Bruceton data on the assumption of a logistically distributed population. The tabular (and graphed) values of $\delta$, G, and H are asymptotic values; that is, they were computed using large-sample statistical theory and taking into account the properties of the Bruceton data collection method. The approach is patterned after the original work done by the Applied Mathematics Panel assuming the Gaussian distribution (references 1 and 2).

29. The computations of the 50% point should be the same for either distribution, since the assumption that the distributions are symmetrical about the mean is valid for both. Consequently, the Bruceton estimate of the mean is:

$$m_B = \left(\frac{\Sigma i n_{x,i}}{\Sigma n_{x,i}} - \frac{1}{2}\right) d + X_0 \tag{23a}$$

or

$$m_B = \left(\frac{\Sigma i n_{o,i}}{\Sigma n_{o,i}} + \frac{1}{2}\right) d + X_0 \tag{23b}$$

These are equivalent to equations 13a and b of the previous section:

$$\bar{X} = X_0 + d\left(\frac{A}{N} - \frac{1}{2}\right), \quad \text{(for the fires), or} \tag{13a}$$

$$\bar{X} = X_0 + d\left(\frac{A}{N} + \frac{1}{2}\right), \quad \text{(for the fails)} . \tag{13b}$$

30. The intermediate parameter, M, for estimating variability parameters was defined in the same way for the logistic Bruceton as for the Gaussian:

$$M = \frac{\left(\Sigma n_i\right)\left(\Sigma i^2 n_i\right) - \left(\Sigma i n_i\right)^2}{\left(\Sigma n_i\right)^2} \tag{24}$$

Where, in the asymptotic case, either $n_{x,i}$ or $n_{x,o}$ can be used for $n_i$.

This is equivalent to equation 14 in the previous section.

$$M = \frac{NB - A^2}{N^2} \tag{14}$$

The value of g (the estimate of $\gamma$) would be found by applying equation 16 followed by equation 17:

$$E = M + \delta \tag{16}$$

where $\delta = 0$ when $0.65 < M < 10$; and

$$g = E \cdot d. \tag{17}$$

31. The variance of $\bar{x}$, $(V_{\bar{x}})$, and g, $(V_g)$, can be estimated using the method of maximum likelihood[6]. The likelihood function is

$$P = \prod_{1}^{N_x} p_i \prod_{1}^{N_o} q_i \tag{25}$$

and its logarithm

$$L = \sum^{N_x} \ln p_i + \sum^{N_o} \ln q_i \tag{26}$$

where $N_x$ is the number of responses at all levels, and $N_o$ is the number of non-responses at all levels. Maximum likelihood theory gives the asymptotic variance of $\bar{x}$ and g as

$$V_{\bar{x}} = -\frac{1}{\mathcal{E}\left(\frac{\partial^2 L}{\partial \bar{x}^2}\right)}, \text{ and} \tag{27}$$

$$V_g = -\frac{1}{\mathcal{E}\left(\frac{\partial^2 L}{\partial \gamma^2}\right)} \tag{28}$$

The symbol $\epsilon$ is used to indicate the expected values of the derivatives. The expected values for these derivatives can be evaluated to give

$$V_{\bar{x}} = \frac{\gamma^2}{\Sigma p_i^2 q_i^2}\text{ , and} \tag{29}$$

$$V_g = \frac{\gamma^2}{\Sigma p_i^2 q_i^2 y_i^2}\text{ , where } y_i \text{ is defined as} \tag{30}$$

$$y_i = \frac{x_i - u}{\gamma} \quad \text{and}$$

where the summation is over all trials at a level as well as over all levels of the Bruceton test. Since for the usual Bruceton notation the summation over i means a summation only over the levels, the above equations must be rewritten for consistency; doing this and solving for the standard deviation gives

$$s_m = s_{\bar{x}} = \sqrt{V_{\bar{x}}} = \frac{\gamma}{\sqrt{\Sigma(n_{x,i} + n_{o,i})p_i q_i}}\text{ , and} \tag{31}$$

$$s_g = \sqrt{V_g} = \frac{\gamma}{\sqrt{\Sigma(n_{x,i} + n_{o,i})p_i q_i y_i^2}}\text{ , where} \tag{32}$$

$s_m$ and $s_{\bar{x}}$ are used interchangeably.

As with the case in which the normal distribution is assumed, we should be able to express the values of $s_m$ and $s_\gamma$ as

$$s_m = \frac{G\gamma}{\sqrt{N}}\text{ , and} \tag{33}$$

$$s_\gamma = \frac{H\gamma}{\sqrt{N}}\text{ .} \tag{34}$$

Here N is the smaller of the two values $N_x$ and $N_o$, but in the asymptotic case $N_x = N_o$, so that either one can be used. Solving

(31) and (33) for G, and solving (32) and (34) for H, gives

$$G = \sqrt{\frac{N}{\Sigma(n_i + m_i)p_i q_i}}, \text{ and} \tag{35}$$

$$H = \sqrt{\frac{N}{\Sigma(n_{x,i} + n_{o,i})p_i q_i y_i^2}} . \tag{36}$$

The parameters G and H would then be used in equations 18 and 19 to compute $s_m$ and $s_g$:

$$s_m = (G \cdot g)/\sqrt{N} \tag{18}$$

and

$$s_g = (H \cdot g)/\sqrt{N} . \tag{19}$$

32. We derived numerical values for the correction terms δ, G, and H by generating expected very-large-sample Bruceton results for various combinations of step size and spacings of the mean from the closest test level. By expressing the step size in units of the population γ, and the spacing of the mean in units of step size, we generated dimensionless variables which are independent of the actual distribution μ and γ. From the expected Bruceton results for each combination of step size and mean spacings δ, G, and H were computed using equations 15, 17, 24, 35 and 36.

33. Arbitrarily we chose μ to be 1000 and γ to be 10. We then chose step sizes ranging from d = 0.1γ to d = 10γ. For each of these center levels we then set up an array of levels $X_i$ (spaced d units apart) above and below the center level and including the center level. For each of the levels in the array we then computed the logit value, of $y_i$, associated with $X_i$ by

$$y_i = \frac{X_i - 1000}{10}$$

The expected $P_i$ and $q_i$ for each level was computed from $y_i$ by equation 5. We then chose arbitrarily a level several steps below the mean. At this bottom level, $X_i$, we stipulated that there would

be one trial which would be a fail.* (As will be seen, this level must be chosen far enough below the mean that the probability of fire is quite small.) The expected number of fires at the next higher level is equal to the number of fails at the previous level.** That is, for all levels we can say that

$$n_{x,(i+1)} = n_{o,i} \ . \tag{37}$$

At any level above the lowest the expected number of fails will depend on the probability of firing at the level and the expected number of fires as generated by equation (37). The expected number of fails is

$$n_{o,(i+1)} = \frac{n_{x,(i+1)} \hat{q}_{(i+1)}}{\hat{p}_{(i+1)}} \ ; \tag{38}$$

in our work $n_o$ was rounded off to the nearest whole number. If the Bruceton test size was less than 100,000,000 the process was repeated starting at lower levels. Also the process was iterated similarly until successive values of $\delta$, G, and H showed no changes in the first four significant figures.

34. A specific computation is included to show the process:

We remember that $\mu$ = 1000 and $\gamma$ = 10.
We wish to generate the values for the case when the step size is 40, that is, when $\gamma/d$ = 0.4; and when the central level (the level closest to the mean) is 0.4 of a step away from $\mu$. The central level is therefore, 1016.
We assign i = 0 to the central level and then set up an array of levels and indices below and above x. The levels are 40 units apart: 896, 936, 976, 1016, 1056, 1096; the corresponding indices are -3, -2, -1, 0, +1, +2.

---

*We could have started with two or more fails (and no fires) at the bottom level without changing the results provided the probability was small enough.

**There are certain properties of the Bruceton data collection plan which can be seen by inspection. First we define a closed Bruceton test as one in which the test at the last level would lead to the same level as the one at which the test began. For any closed Bruceton test the total number of fires is equal to the total number of fails; also, the number of fails at any level is equal to the number of fires at the next higher level. If the test is not closed, the number of fails at one level will differ from the number of fires at the next higher level by no more than 1. However, with a large number of trials at all levels except the top and bottom an assumption of equality will introduce negligible error.

At each level, $x_i$, the values of $y_i$, $\hat{p}_i$, and $\hat{q}_i$ are computed.

Using the algorithm of equations 37 and 38 and starting at $i = -3$ the values of $n_{x,i}$ and $n_{o,i}$ are computed at each level, yielding the results given in Table 4.

The various summations overall levels are computed:

$$\sum n_{x,i},\ \sum i n_{x,i},\ \sum i^2 n_{x,i},\ \sum (n_{x,i} + n_{o,i})\hat{p}_i\hat{q}_i,$$

$$\sum (n_{x,i} + n_{o,i})\hat{p}_i\hat{q}_i y_i^2.$$

The values of M, G, and H are computed using equations 24,35, and 36.

When the lowest level is set at 856 rather than 896 an eight-level pattern is generated. The values of the summations and M, G, and H for this latter case are also shown in Table 4. There are differences between the six-level and eight-level results, the greatest difference being less than 0.05%. Since the eight-level pattern results were the same as larger-level patterns they can be used as the asymptotic values.

35. The process described above was carried out for values of $\gamma/d$ ranging from 0.1 to 10.0. Even though the algorithm would work outside of the above range we elected not to go further because we see no practical use for the results. In fact, either with such extremely large steps ($\gamma/d < 0.1$) or with extremely small steps ($\gamma/d > 10.0$) a Bruceton test would be of dubious value. For values of $\gamma/d$ from 0.65 to 10.0, M was equal to $\gamma/d$ to six significant figures and independent of D (the spacing between the central level and $\mu$). For $\gamma/d$ running from 0.1 to 0.60, calculations were made for values of D of 0.0, 0.1, 0.2, 0.3, 0.4, and 0.5. G and H were similarly dependent upon D for $\gamma/d$ below approximately 0.65 and otherwise independent of D. The asymptotic values of M, G, and H are plotted in Figures 7, 8, and 9.

36. These graphs should not be used to find values for use in Bruceton computations, because they have been made to show the complete range of values and to show the shapes of the functions. Detailed plots can be found in Appendix A. There are some other differences between the two sets of graphs. The asymptotic graphs (Figures 7, 8, and 9) are of the various parameters versus $\gamma/d$. But of course in an actual Bruceton test we cannot know $\gamma$; we can compute x from equation 13, M from equation 14, and D from equation 15. The variable E has been defined as the ratio of the expected value of g to the step size, d (equation 17). When M is less than 0.65 it will be a function of both E and D as can be seen from the following typical values:

TABLE 4 TYPICAL CALCULATIONS OF ASYMPTOTIC VALUES FOR M, G, & H

| $i$ | $x_i$ | $y_i$ | $p_i$ | $n_{x,i}$ | $n_{o,i}$ |
|---|---|---|---|---|---|
| 2 | 1096 | 9.6 | 0.999932 | 5 | 0 |
| 1 | 1056 | 5.6 | 0.996316 | 1339 | 5 |
| 0 | 1016 | 1.6 | 0.832018 | 6634 | 1339 |
| -1 | 976 | -2.4 | 0.0831727 | 602 | 6634 |
| -2 | 936 | -6.4 | 0.0016588 | 1 | 602 |
| -3 | 896 | -10.4 | 0.0000304316 | 0 | 1 |

| | six levels | eight levels* |
|---|---|---|
| $\sum n_{x,i}$ | = 8,581 | 281,984,179 |
| $\sum i n_{x,i}$ | = 745 | 24,496,639 |
| $\sum i^2 n_{x,i}$ | = 1,965 | 64,572,203 |
| $\sum (n_{x,i} + n_{o,i})\hat{p}_i\hat{q}_i$ | = 1,672.05 | 54,946,457 |
| $\sum (n_{x,i} + n_{o,i})\hat{p}_i\hat{q}_i y_i^2$ | = 6,226.62 | 204,614,821 |

| | six levels | eight levels* |
|---|---|---|
| M | = 0.221456 | 0.221455 |
| G | = 2.265092 | 2.265388 |
| H | = 1.173932 | 1.173439 |

*See explaination in latter part of paragraph 34.

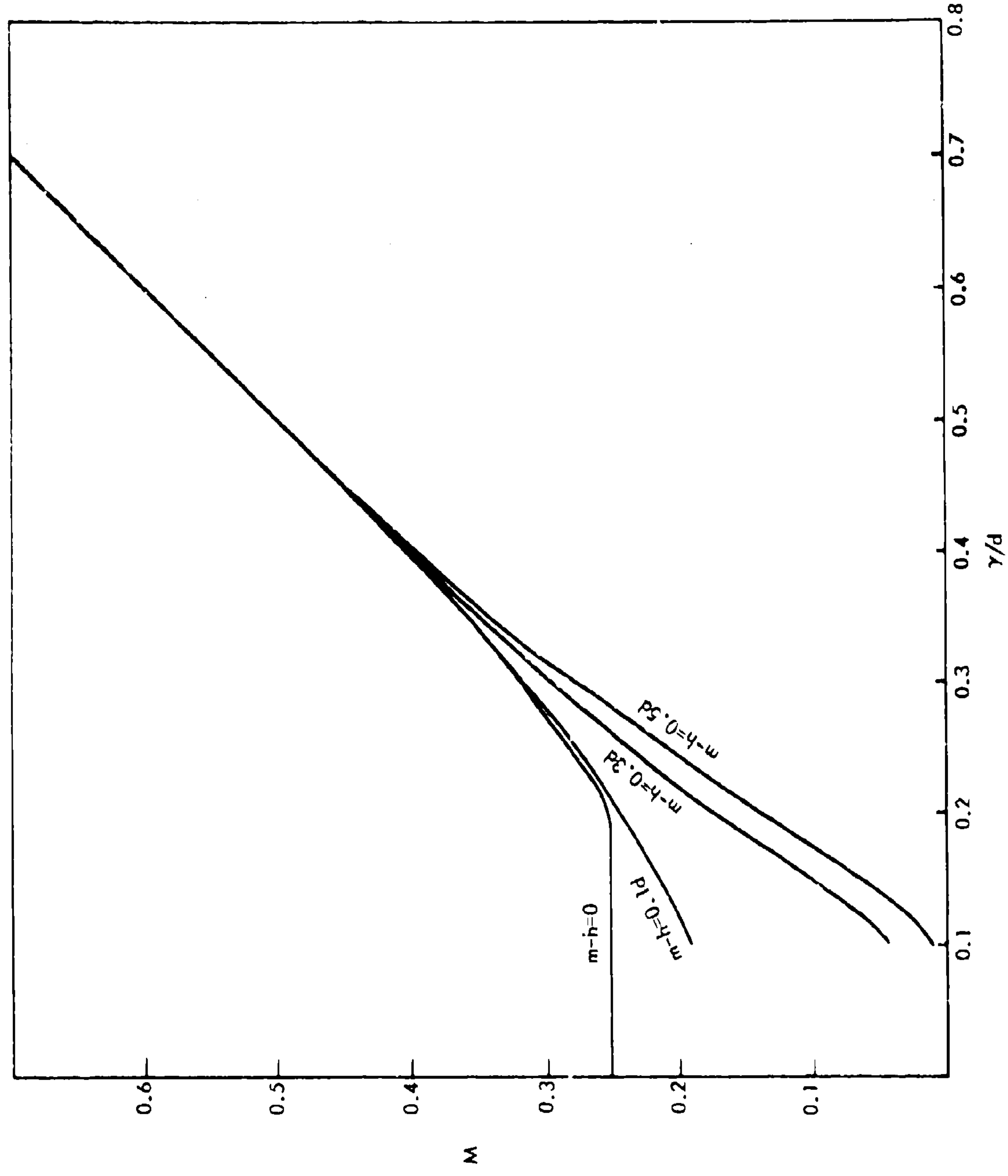


FIG. 7 M FOR LOGISTIC BRUCETON

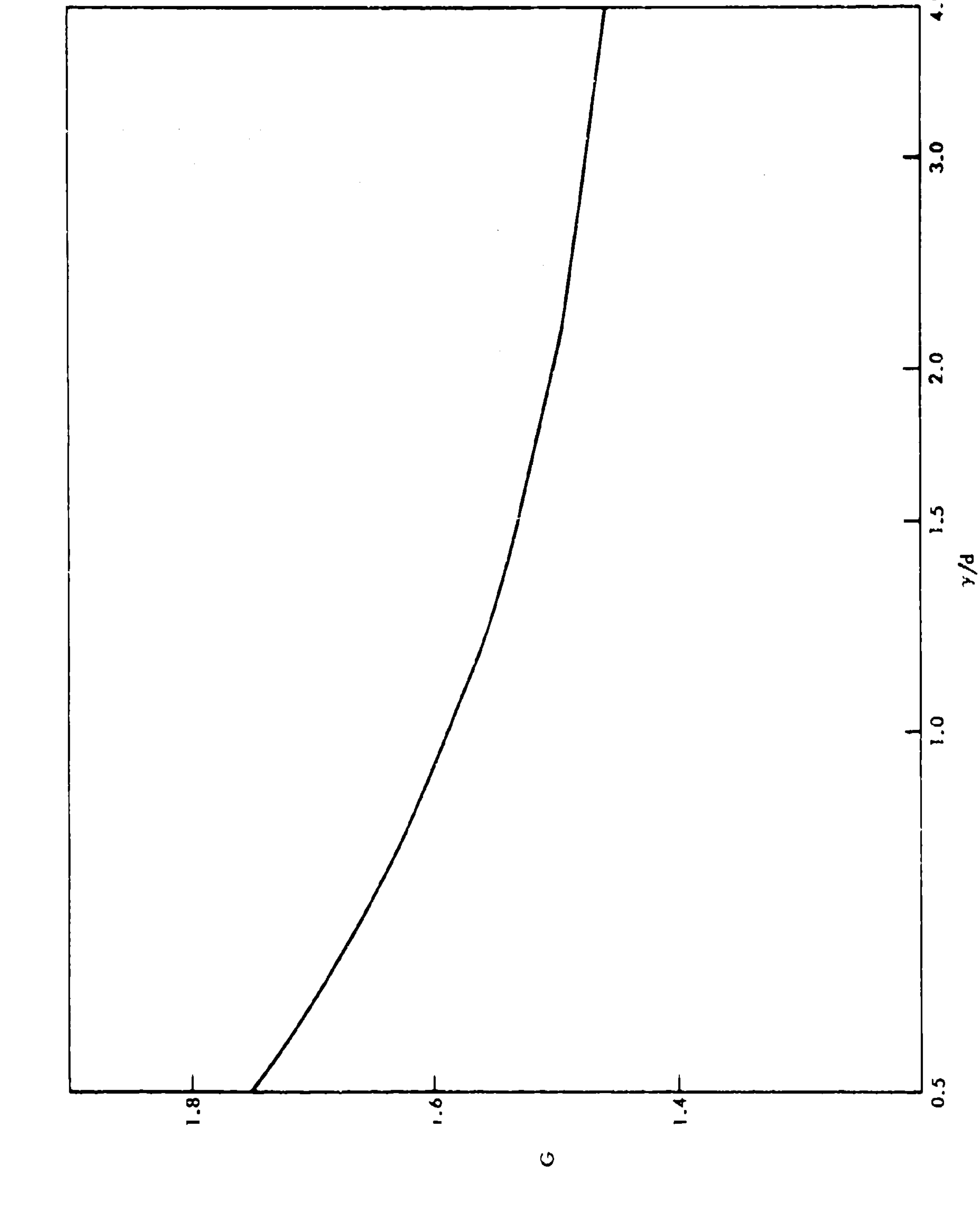


FIG. 8 G FOR LOGISTIC BRUCETON

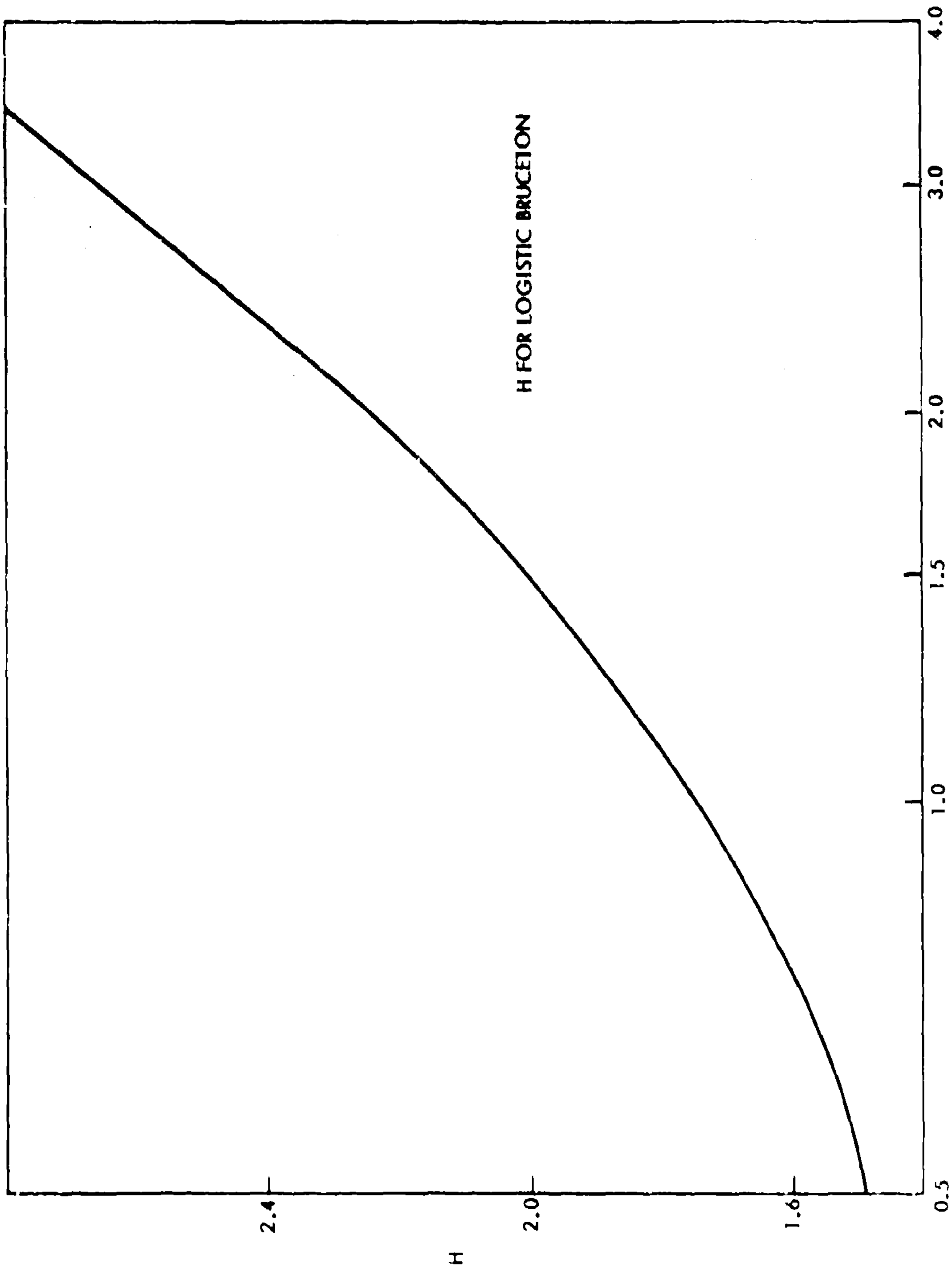


FIG. 9 H FOR LOGISTIC BRUCETON

| D \ E | 0.240 | 0.250 | 0.260 |
|---|---|---|---|
| 0.00 | 0.2806 | 0.2860 | 0.2919 |
| 0.10 | 0.2724 | 0.2789 | 0.2856 |
| 0.20 | 0.2517 | 0.2605 | 0.2694 |
| 0.30 | 0.2271 | 0.2386 | 0.2500 |
| 0.40 | 0.2080 | 0.2214 | 0.2346 |
| 0.50 | 0.2058 | 0.2150 | 0.2289 |

However during an ordinary Bruceton data analysis it is necessary to derive E from the value of M obtained from equation (14). By a four-term Lagrangian interpolation the numbers can be reorganized into a table of E values:

| D \ M | 0.200 | 0.220 | 0.240 | 0.260 | 0.280 | 0.300 |
|---|---|---|---|---|---|---|
| 0.00 | - | - | - | 0.1890 | 0.2588 | 0.2728 |
| 0.10 | .1037 | 0.1352 | 0.1787 | 0.2189 | 0.2517 | 0.2800 |
| 0.20 | 0.1810 | 0.2036 | 0.2265 | 0.2494 | 0.2718 | 0.2937 |
| 0.30 | 0.2170 | 0.2339 | 0.2512 | 0.2689 | 0.2871 | 0.3056 |
| 0.40 | 0.2342 | 0.2489 | 0.2642 | 0.2799 | 0.2962 | 0.3130 |
| 0.50 | 0.2394 | 0.2536 | 0.2682 | 0.2834 | 0.2991 | 0.3152 |

We found that a better graphical presentation could be made by plotting $\delta = f(M,D)$ rather than $E = f(M,D)$ where $E = M + \delta$. The values of $\delta$ which correspond to the E values above are:

| D \ M | 0.200 | 0.220 | 0.240 | 0.260 | 0.280 | 0.300 |
|---|---|---|---|---|---|---|
| 0.00 | - | - | - | -0.0710 | -0.0212 | -0.0272 |
| 0.10 | -0.0963 | -0.0848 | -0.0613 | -0.0411 | -0.0283 | -0.0200 |
| 0.20 | -0.0190 | -0.0164 | -0.0135 | -0.0106 | -0.0082 | -0.0063 |
| 0.30 | +0.0170 | +0.0139 | +0.0112 | +0.0089 | +0.0071 | +0.0056 |
| 0.40 | +0.0342 | +0.0289 | +0.0242 | +0.0199 | +0.0162 | +0.0130 |
| 0.50 | +0.0394 | +0.0336 | +0.0282 | +0.0234 | +0.0191 | +0.0152 |

The tabular and graphic values of G and H are given as functions of E and D rather than of M and D because, once E is found, it is the best available estimate of $\gamma/d$ and should be used as such.

## V. MONTE CARLO STUDIES

37. In order to check the results of the theoretical work a Monte Carlo investigation was carried out. A program for the IBM 7090 computer was prepared to generate random numbers with a logistic distribution to represent the sensitivities or critical levels of

the test items. The distribution parameters, $\mu$ and $\gamma$, and the sample size, $N_s$, were adjustable. A run of one hundred Bruceton tests was made for each combination of the quantities investigated. The sample mean, $m_s$, and the variability, $g_s$, were computed for each test. The starting level was fixed at 20.0 and the step size at 1.0. For each Bruceton test the estimates $m_B$ and M as defined in the previous section, equations 23 and 24, were obtained. (Subscript B indicates an estimate obtained from a Bruceton test and subscript s refers to a value for the sample.) For each run the average and standard deviation of each of the quantities $m_B$ and M was found.

38. We showed in the previous section that g/d can be taken as equal to M if M > 0.65. For this computer study (and for that matter in ordinary testing), $\gamma/d$ can be taken as equal to M if M > 0.4. It was convenient to use this relationship for the Monte Carlo studies. Since the step size, d, was chosen as unity we can therefore take M, in this range, as equal to $g_B$, the Bruceton estimate of $g_s$. In this case, then, the average and standard deviation of M will be the average and standard deviation of $\gamma_B$. The computation of $g_s$ when $\gamma/d < 0.4$ depends upon the relative position of the mean and the nearest step level. For each run we know this for the population mean. If the sample mean were the same as the population mean we could use the curves of figure 7 to estimate $\gamma_B$ from the value of M. Inspection of the curves of figure 7 shows that the relative position of the mean is not as critical when the mean is midway between step levels as when it is on a step level. For the runs in which the population mean was midway between step levels the values of the mean and standard deviation of $g_B$ were estimated from the mean and standard deviation of M by using the curve of figure 7 for the case with this position of the mean. The values of the mean and standard deviation of $g_B$ given in the tables of Appendix C for $\gamma/d$ less than 0.4 were obtained in this way. Since d = 1, the value of $\gamma/d$ is to be found in the second column, headed by $\gamma$. These values of $g_B$ are too large and their standard deviations are too small since the means of the individual tests do not fall midway between the step levels. (This can be seen by comparing the results which would be obtained from the curve for m - h = 0.3 with the results obtained from the assumption that the mean is midway between levels.) We did not estimate the mean and standard deviation of $g_B$ for tests with $\gamma/d$ less than 0.4 and with the population mean on a step level.

39. For each run of 100 trials the average and standard deviation of the observed $m_B$ and $g_B$ were calculated. But the standard deviation of $m_B$ is the observed $s_m$, and similarly the standard deviation of $g_B$ is $s_g$. Substituting $s_m$ and $s_g$ into equations 33 and 34, using the average $g_B$ for $\gamma$ and setting N (the number of fires or fails) equal to one half of $N_s$ permits us to find values of G and H which are representative of the experimental conditions. This is because the experiments must use observed values since we cannot know the population parameters.

40. Since theoretical development in the previous section assumes large samples the first interest in the Monte Carlo investigation was the effect of sample size on these results. Runs were therefore made with samples as small as ten and as large as five hundred. It is not difficult to see that for small samples a few unexpected results in the early part of the test could influence the estimates of the mean and the dispersion unduly. This would be especially the case if the step were small or, in other words, if the value of gamma were large in comparison with the step size. In our program the step size was always one. The effect of the relation of step size to the value of gamma was investigated by using population values of gamma from 0.1 to 100. ($d/\gamma$ from 10 to 0.01) The effect of choice of starting level was investigated by using population means of 19.5, 20.0, 20.5, and 25.0. Since the starting level was always at 20.0 these gave tests in which the starting level was one half-step above, on, one half-step below, or five steps below the mean. The results obtained are summarized in the tables of Appendix C.

## Bias in the Estimate of Gamma

41. Other Monte Carlo investigations have shown that the Bruceton test, based upon the assumption of a normal distribution, gives a biased estimate of the standard deviation (references 7, 8). It is evident from our results that a similar bias exists in the estimate of gamma for the logistic distribution. The amount of this bias is shown in the last column of tables C1 to C20 as the ratio of $g_s$ to $g_B$. Tables 5 and 6 are condensations of these results. We observe that: the bias decreases as the sample size is increased; the bias is greater as the ratio $\gamma/d$ increases, that is, as the step becomes small with respect to the value of gamma. Our results do not show a bias when $N_s = 20$ with starting level at the mean and the ratio $\gamma/d$ equal to or greater than one (medium or small step size). This should not be taken as a recommendation for a test plan since uncertainty in the knowledge of the value of the mean would make it impossible to plan a test so that the mean would be on a test level. It is a temptation to use the bias ratio, $g_s/g_B$, as a fudge factor to try to improve the estimate of $\gamma$. We counsel strongly against such a measure because the individual $g_B$ values are scattered so greatly about their average.

## Effect of Sample Size on Estimates of G and H

42. The values of G and H as obtained from the Monte Carlo investigations agree well with the values predicted by the theoretical computations when the sample size was large (five hundred items) and the step size was not extremely large or small. The computation of the values of G and H was based upon the values of gamma obtained by the Bruceton analysis. For samples of less than five hundred the Monte Carlo values of G and H were greater than the theoretical values. Part, but not all, of this variation is due to the fact that the value of gamma is underestimated for these smaller samples. The Monte

Table 5 Bias in Estimate of Gamma

| $N_s$ \ $\gamma/\sigma$ | 0.5 | 1.0 | 2.0 | 3.0 |
|---|---|---|---|---|
| 0 | 1.2015 | 1.3300 | 1.7119 | 1.9870 |
|  | 1.1954 | 0.9992 | 0.9965 | 1.0026 |
|  | 1.1099 | 1.4242 | 1.8175 | 1.9508 |
| 50 | 1.0828 | 1.1754 | 1.2577 | 1.3794 |
|  | 1.0881 | 1.1185 | 1.2921 | 1.4253 |
|  | 1.0255 | 1.1252 | 1.2274 | 1.3780 |
| 100 | 1.0402 | 1.0533 | 1.1411 | 1.1933 |
|  | 1.0914 | 1.0634 | 1.1860 | 1.2209 |
|  | 1.0476 | 1.0667 | 1.1071 | 1.1979 |
| 500 | 0.9988 | 0.9871 | 1.0068 | 1.0330 |
|  | 0.9920 | 1.0252 | 1.0498 | 1.0367 |
|  | 0.9914 | 1.0198 | 1.0302 | 1.0406 |

Upper figure
$\mu = 19.5$
$\mu = 20.0$
$\mu = 20.5$

Table 6 Bias in Estimate of Gamma for Sample Size of Twenty

| $\gamma/\sigma$ | $\mu=19.5$ | $\mu=20.0$ | $\mu=20.5$ | $\mu=25.0$ |
|---|---|---|---|---|
| 0.5 | 1.2015 | 1.1954 | 1.1099 | 1.1392 |
| 0.7 | 1.2915 | 1.2603 | 1.2267 | 1.1288 |
| 0.85 | 1.3939 | 1.2124 | 1.1624 | 1.1327 |
| 1.0 | 1.3300 | 0.9992 | 1.4242 | 1.0028 |
| 1.5 | 1.7225 | 0.9982 | 1.6289 | 1.0114 |
| 2.0 | 1.7119 | 0.9965 | 1.5175 | 1.0211 |
| 2.5 | 2.0594 | 0.9989 | 2.0226 | 1.1251 |
| 3.0 | 1.9870 | 1.0026 | 1.9508 | 1.4552 |
| 4.0 | 2.4999 | 1.0050 | 2.5306 | 1.3526 |

Carlo investigations indicate that values of $s_m$ and $s_g$ will be seriously underestimated when the formulas based on the assumption of large samples are used with smaller samples.

43. As an example of this underestimation take the results for a sample size of 50 with the population mean at 20.5, population gamma at 1, as given in Table C-11. In this case the average estimate of gamma obtained from the one hundred tests was 0.89. From Tables A-2 and A-3, we would then get 1.585 and 1.745 as values of G and H for this value of gamma. Using equations (18) and (19) with N = 25 (half of the sample), we would obtain $s_m = 0.282$ and $s_g = 0.311$. Table C-11 gives $s_m = 0.274$ and $s_g = 0.346$ so that we have reasonable agreement for $s_m$, but the value of $s_g$ as obtained from use of Table A-3 is too small.

44. Tables 7 and 8 give the effect of sample size on the values of G and H as computed from the Monte Carlo investigations. The theoretical values are given under the infinite sample size entry. Since there is no reason to expect the values of G and H to be affected by the position of the mean when the ratio $\gamma/d$ is greater than 0.4, the observed differences between values for the same sample size and $\gamma/d$ ratio can be considered to be due to random error. The three values in the tables were therefore arrayed. It can be seen that within the estimate of random error noted above the values of G for larger samples agree with the theoretical values. As the sample size decreases the value of G increases. The values of H also agree well for large samples. However, for values of $\gamma/d$ of 1.0 and 2.0, the value of H decreases for small samples. This is due to the fact that in these cases the step is so small that the test does not have time to cover the full range of levels before the sample is exhausted. The result is, therefore, that an artificially precise, but quite inaccurate, determination is made of gamma.

## Effect of Starting Level

45. The effect of a poor choice of starting level was investigated by Monte Carlo runs in which the mean of the population was at 25. The starting point of the test was kept at 20.0. As might be expected, the value of the mean as computed from the Bruceton tests tended to be near the starting point when a small number of items was tested and also for a large value of the ratio $\gamma/d$, that is, when the distribution function was relatively flat. When a small number of items is tested the estimate of the dispersion parameter is too small and shows more variation from sample to sample than would be expected from the large sample theory. The results are given in Tables C-4, C-8, C-12, C-16, and C-20. The comparative effect on the mean is shown for three step sizes in Table 9 which gives the values of $(u - m_B)/\gamma$ and the corresponding true percent response of the estimated fifty percent point. The effect on the value of gamma is shown in Tables 10 and 11. Table 10 gives the values of the bias factor $g_s/g_B$. Comparison with the values of this bias found in tests with better choice of starting point shows little difference. Table 11 gives the values of H, which is a measure of the variability of the estimate of gamma. It will be seen that for small samples these are larger than when a good choice

Table 7 Effect of Sample Size on value of G

$\gamma/d = 0.7$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$=20.5 | average |
|---|---|---|---|---|
| 10 | 2.6980 | 2.5871 | 2.7994 | 2.6948 |
| 20 | 2.4133 | 1.9499 | 2.0331 | 2.1321 |
| 50 | 1.7944 | 1.7714 | 1.9956 | 1.8538 |
| 100 | 1.6856 | 1.5412 | 1.8502 | 1.6923 |
| 500 | 1.9019 | 1.5720 | 1.4675 | 1.6471 |
| Infinite | - | - | - | 1.657 |

$\gamma/d = 1.0$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$=20.5 | average |
|---|---|---|---|---|
| 10 | 3.4533 | 2.6696 | 3.2983 | 3.1404 |
| 20 | 2.1599 | 2.1014 | 2.0653 | 2.1089 |
| 50 | 1.8659 | 1.7569 | 1.5375 | 1.7201 |
| 100 | 1.7452 | 1.7947 | 1.7345 | 1.7581 |
| 500 | 1.3624 | 1.7222 | 1.4635 | 1.5160 |
| Infinite | - | - | - | 1.585 |

$\gamma/d = 2.0$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$=20.5 | average |
|---|---|---|---|---|
| 10 | 3.1579 | 3.5380 | 3.6335 | 3.4431 |
| 20 | 2.5096 | 2.7503 | 2.3119 | 2.5239 |
| 50 | 1.7646 | 1.7569 | 1.5577 | 1.6931 |
| 100 | 1.6388 | 1.6205 | 1.4989 | 1.5861 |
| 500 | 1.5508 | 1.4057 | 1.5202 | 1.4922 |
| Infinite | - | - | - | 1.501 |

Table 8 Effect of Sample Size on value of H

$\gamma/d = 0.7$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$=20.5 | average |
|---|---|---|---|---|
| 10 | 2.4122 | 1.7636 | 1.4802 | 1.8853 |
| 20 | 1.7724 | 1.3645 | 2.0666 | 1.7345 |
| 50 | 1.7218 | 1.6561 | 1.7294 | 1.7024 |
| 100 | 1.4804 | 1.7171 | 1.4852 | 1.5609 |
| 500 | 1.5743 | 1.5356 | 1.6419 | 1.5839 |
| Infinite | - | - | - | 1.577 |

$\gamma/d = 1.0$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$=20.5 | average |
|---|---|---|---|---|
| 10 | 1.9087 | 1.5850 | 1.7838 | 1.7591 |
| 20 | 1.7603 | 1.7916 | 1.5260 | 1.6926 |
| 50 | 1.6482 | 1.8939 | 1.9407 | 1.8276 |
| 100 | 1.4554 | 1.7629 | 1.7095 | 1.6426 |
| 500 | 1.6723 | 1.9354 | 1.8860 | 1.8312 |
| Infinite | - | - | - | 1.745 |

$\gamma/d = 2.0$

| Sample | $\mu$=19.5 | $\mu$=20.0 | $\mu$-20.5 | average |
|---|---|---|---|---|
| 10 | 2.3629 | 1.6225 | 1.6173 | 1.8676 |
| 20 | 2.0225 | 1.6898 | 1.7771 | 1.8298 |
| 50 | 2.0313 | 1.8939 | 2.0827 | 2.0026 |
| 100 | 2.0174 | 2.1640 | 2.1115 | 2.0976 |
| 500 | 2.1511 | 2.3735 | 2.1579 | 2.2275 |
| Infinite | - | - | - | 2.240 |

of starting level was made. Similar effects were observed in Monte Carlo tests with populations having a normal distribution. (Reference 8).

### Table 9

### Effect of Poor Starting Point on Determination of the Mean

(Table gives values of $(\mu - m_B)/\gamma$ below the line and the corresponding true percent point above the line for $\gamma/d$ values of 0.7, 1.0, and 2.0)

| $\gamma/d$ | | 0.7 | 1.0 | 2.0 |
|---|---|---|---|---|
| Starting percent point | | 0.10 | 0.67 | 7.60 |
| Sample Size | 10 | 0.650 / 34.31 | 0.703 / 33.12 | 0.725 / 32.64 |
| | 20 | 0.087 / 47.83 | 0.181 / 45.49 | 0.299 / 42.58 |
| | 50 | 0.070 / 48.25 | 0.068 / 48.30 | 0.111 / 47.23 |
| | 100 | 0. / 50.00 | 0.075 / 48.13 | 0.065 / 48.38 |
| | 500 | 0. / 50.00 | 0.015 / 49.63 | 0.016 / 49.60 |

Table 10

Effect of Poor Starting Point on the Determination of Gamma

(Values of the Bias Factor $g_s/g_B$)

| Sample Size \ γ/d | 0.7 | 1.0 | 2.0 |
|---|---|---|---|
| 10 | 1.6500 | 1.3934 | 2.4678 |
| 20 | 1.1288 | 1.0028 | 1.0211 |
| 50 | 1.0649 | 0.9842 | 0.9453 |
| 100 | 1.0200 | 1.0162 | 0.9655 |
| 500 | 0.9881 | 1.0104 | 1.0158 |

Table 11

Effect of Poor Starting Point on Value of H

(Values of the Bias Factor $g_s/g_B$)

| Sample Size \ γ/d | 0.7 | 1.0 | 2.0 |
|---|---|---|---|
| 10 | 3.2567 | 3.3538 | 2.9715 |
| 20 | 2.2232 | 2.4274 | 2.4228 |
| 50 | 1.7776 | 1.9905 | 2.0749 |
| 100 | 1.6393 | 1.8627 | 2.2386 |
| 500 | 1.7682 | 1.6963 | 2.0778 |

## Choice of Step Size

46. Some generalizations can be made. The choice of the step size to be used in the test depends upon what is already known about the distribution and upon whether we are more interested in determining the fifty percent point or the value of gamma. The following points should be noted.

a. For large steps (larger than five times gamma) the value of gamma becomes indeterminate if the fifty percent point falls on a test level (Figure 7). For this reason large steps should be avoided unless the position of the fifty percent point is at least approximately known.

b. A small step size would not be good with a test of as few as twenty items unless we can be sure that the starting level is near the population fifty per cent point. For example see the Monte Carlo determination of the fifty percent point for tests of twenty items with the population mean at 25 and a value for gamma of 2.5. Here the starting level is at the 12 percent point of the population and the step size ($d/\gamma$) is 0.4. The average of the Bruceton means was 23.827 which is the 38.5 per cent point of the population.

47. The precision with which the values of the mean and gamma can be measured under ideal conditions for different step sizes can be deduced from the shape of the G and H curves (Figures [illegible]).

a. The values of G show that the most precise measurements of the mean are obtained for small steps. However these ideal conditions are not ordinarily met. As we have just shown a small step gives a poor estimate of the mean when we have a short test with a poor choice of starting level.

b. The minimum value of H occurs when $\gamma/d$ is equal to 0.5 if the mean is on a test level or 0.2 when the mean is midway between test levels. This gives the most precise determination of gamma when the step is two to five times the value of gamma depending upon the position of the mean with respect to the test levels. If we know the mean well enough so that we can be fairly certain that it is nearly halfway between step levels and if we expect to test enough items to obtain a good determination of the response at the two levels nearest the mean a step size as large as five times the expected value of gamma will be best for measuring the dispersion parameter. In this case we would be testing at the estimated 8 and 92 percent points. This would mean that the test should include at least fifty items if we expect to get fairly good estimates of these points. If we cannot test as many as this a smaller step size must be chosen.

## REFERENCES


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# APPENDIX A

Appendix A is a summary of the process for carrying out a logistic Bruceton analysis. Figure A-1 is a flow chart giving the various operations and equations to obtain $\bar{x}$, $g$, $s_m$, and $s_g$. Figures A-2, A-3, and A-4 present graphically the various relationships needed for the computations called out in the flow chart. Tables A-1, A-2, and A-3 from which Figures A-2, A-3, and A-4 were constructed, can be used for greater accuracy. Table A-4 is a compilation of the equations needed to compute any desired response stimulus level and associated one- and two-sided confidence limits. The student t approximation equation, and constants for various percentage levels, are also included in this compilation.

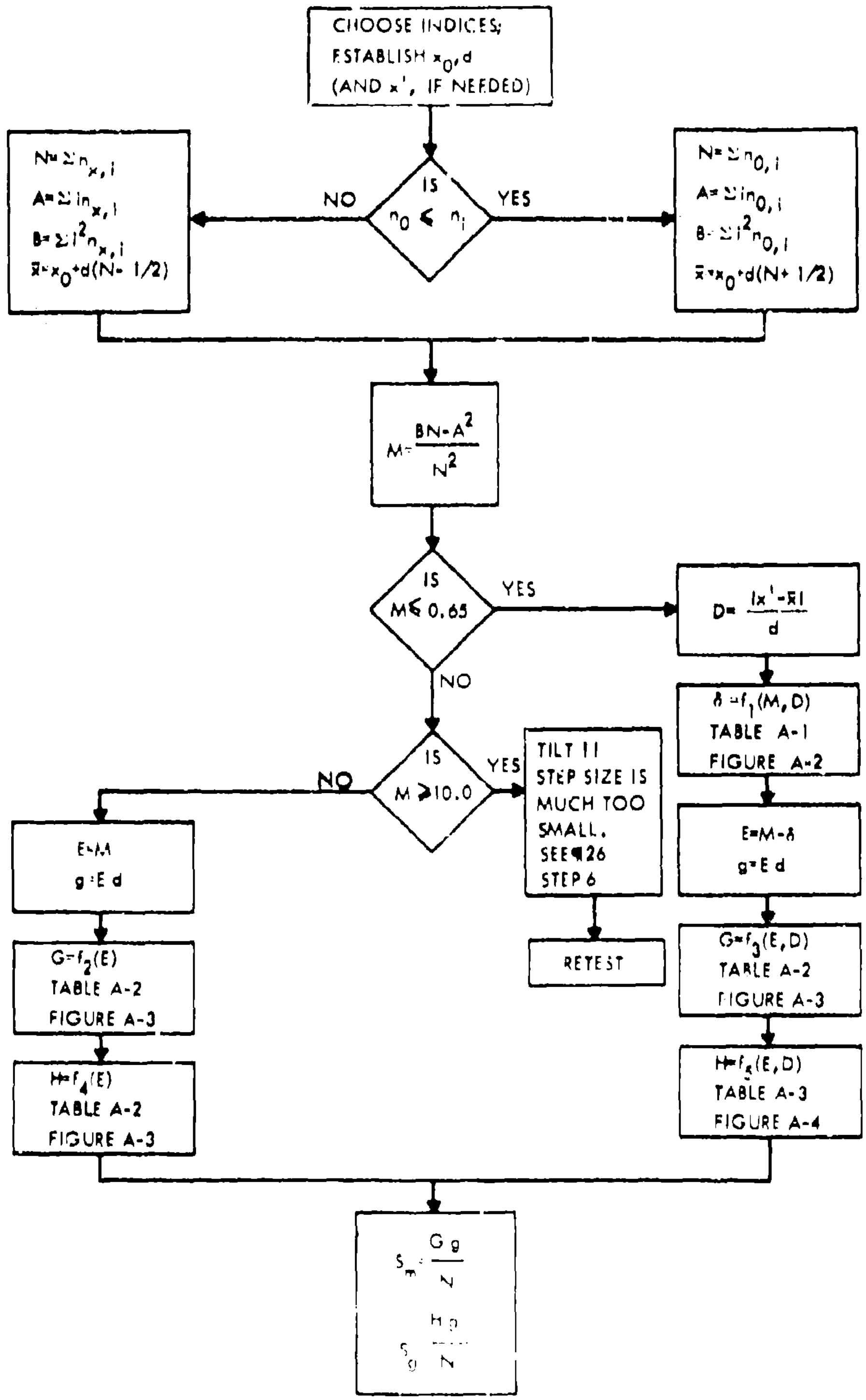


FIG. A-1 FLOW CHART FOR COMPUTATION OF THE LOGIT-BRUCETON PARAMETERS $\bar{x}$, g, $S_m$, AND $S_g$.

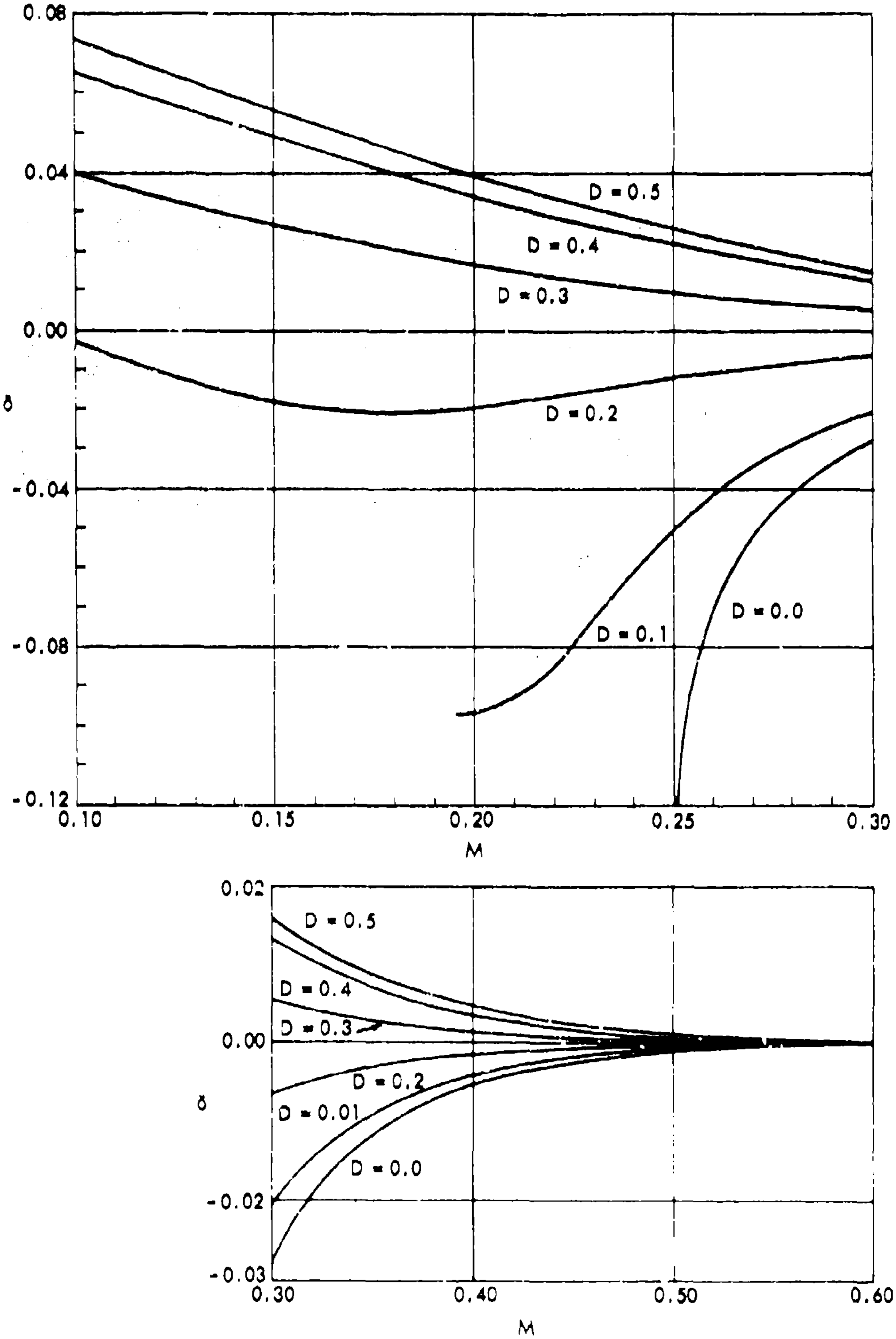


FIG. A-2 VALUES OF δ AS A FUNCTION OF D AND M, FOR $M < 0.60$ TO BE USED IN EQUATION $E = M + \delta$

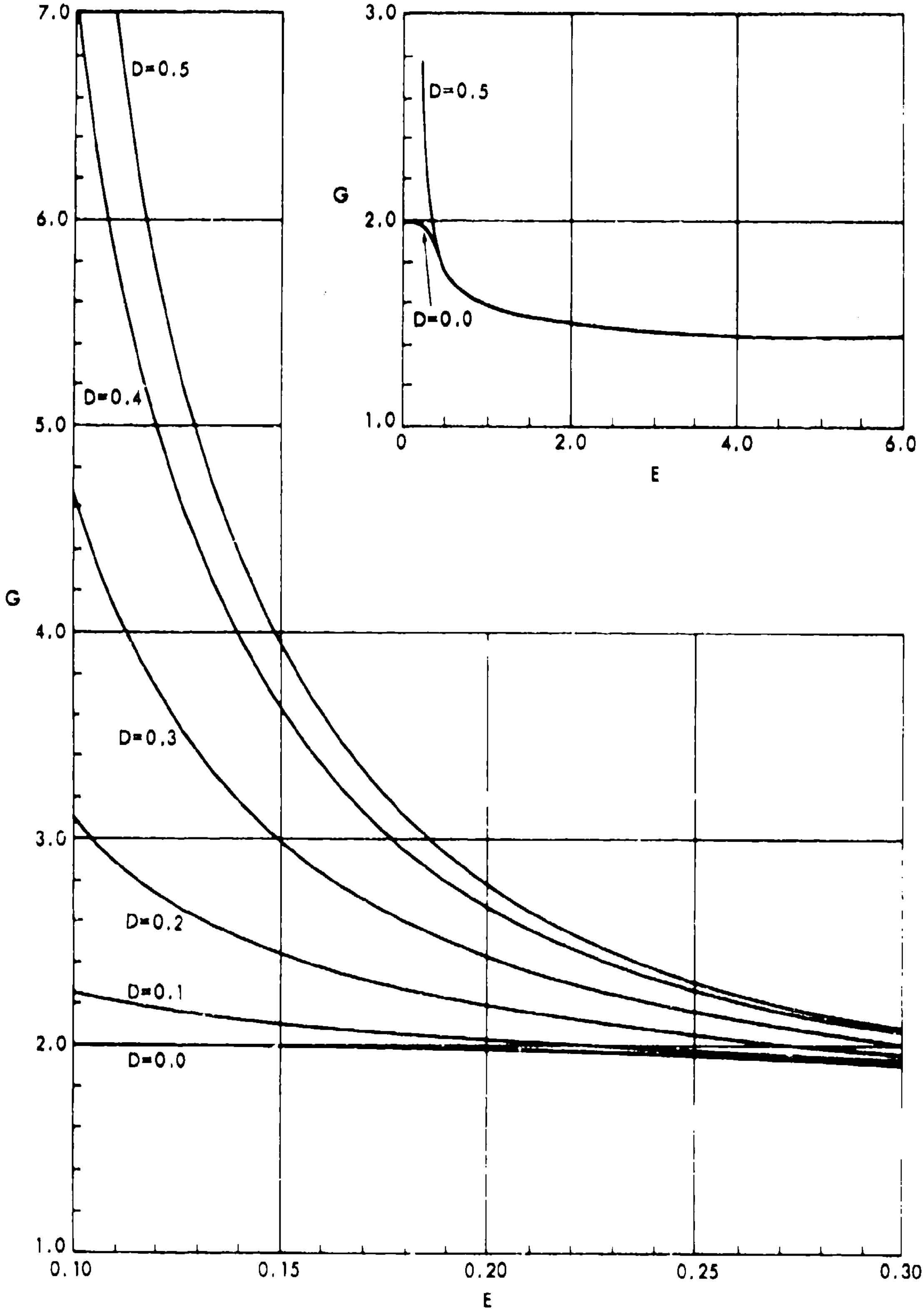


FIG. A-3 VALUES OF G AS A FUNCTION OF D AND E

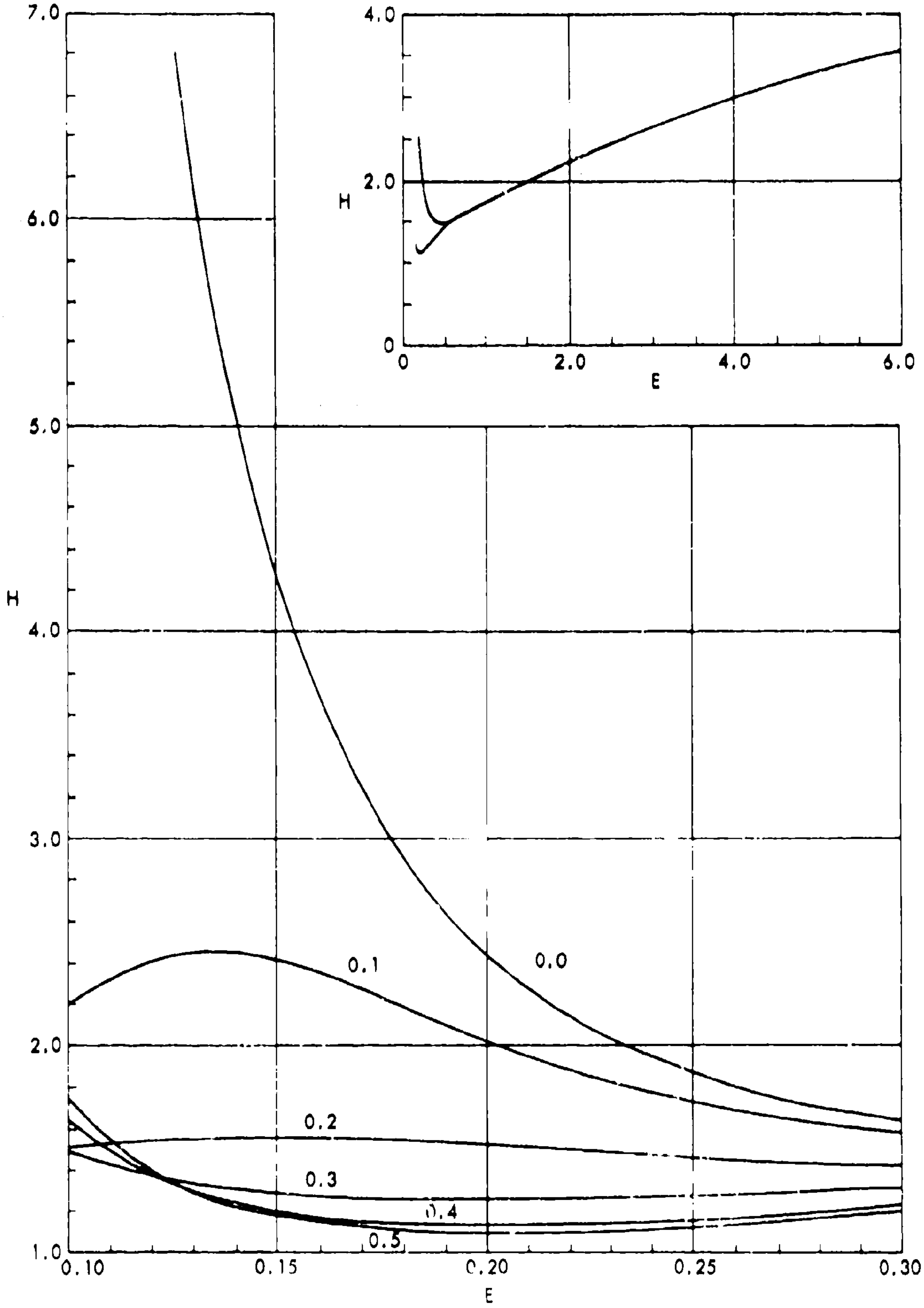


FIG. A-4 VALUES OF H AS A FUNCTION OF D AND E

TABLE A-1 TABULAR VALUES OF THE CORRECTION FACTOR δ

| D ↓ / M → | 0.10 | 0.105 | 0.110 | 0.115 | 0.120 | 0.125 | 0.130 | 0.135 | 0.140 | 0.145 | 0.150 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | - | - | - | - | - | - | - | - | - | - | - |
| 0.1 | - | - | - | - | - | - | - | - | - | - | - |
| 0.2 | -0.0022 | -0.0052 | -0.0071 | -0.0089 | -0.0106 | -0.0122 | -0.0136 | -0.0150 | -0.0162 | -0.0172 | -0.0181 |
| 0.3 | +0.0395 | +0.0381 | +0.0368 | +0.0354 | +0.0341 | +0.0328 | +0.0316 | +0.0304 | +0.0292 | +0.0280 | +0.0268 |
| 0.4 | +0.0650 | +0.0634 | +0.0619 | +0.0603 | +0.0586 | +0.0570 | +0.0554 | +0.0538 | +0.0522 | +0.056 | +0.0490 |
| 0.5 | +0.0729 | +0.0712 | +0.0695 | +0.0678 | +0.0661 | +0.0643 | 0.0626 | +0.0609 | +0.0591 | +0.0574 | +0.0557 |

| D | 0.155 | 0.160 | 0.165 | 0.170 | 0.175 | 0.180 | 0.185 | 0.190 | 0.195 | 0.200 | 0.205 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | - | - | - | - | - | - | - | - | - | - | - |
| 0.1 | - | - | - | - | - | - | - | - | -0.0968 | -0.0963 | -0.0950 |
| 0.2 | -0.0188 | -0.0194 | -0.0198 | -0.0202 | -0.0203 | -0.0203 | -0.0202 | -0.0199 | -0.0195 | -0.0190 | -0.0185 |
| 0.3 | +0.0257 | +0.0246 | +0.0236 | +0.0225 | +0.0215 | +0.0206 | +0.0196 | +0.0187 | +0.0178 | +0.0170 | +0.0162 |
| 0.4 | +0.0475 | +0.0459 | +0.0444 | +0.0428 | +0.0413 | +0.0399 | +0.0384 | +0.0370 | +0.0356 | +0.0342 | +0.0328 |
| 0.5 | +0.0540 | +0.0523 | -0.0506 | +0.0489 | +0.0473 | +0.0456 | +0.0440 | +0.0425 | +0.0409 | +0.0394 | +0.0379 |

| D | 0.210 | 0.215 | 0.220 | 0.225 | 0.230 | 0.235 | 0.240 | 0.245 | 0.250 | 0.251 | 0.252 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | - | - | - | - | - | - | - | - | - | -0.1191 | -0.1070 |
| 0.1 | -0.0927 | -0.0895 | -0.0848 | -0.0791 | -0.0719 | -0.0679 | -0.0613 | -0.0554 | -0.0500 | -0.0490 | -0.0480 |
| 0.2 | -0.0179 | -0.0172 | -0.0164 | -0.0157 | -0.0149 | -0.0142 | -0.0135 | -0.0127 | -0.0119 | -0.0117 | -0.0116 |
| 0.3 | +0.0154 | +0.0146 | +0.0139 | +0.0132 | +0.0125 | +0.0119 | +0.0112 | +0.0106 | +0.0100 | +0.0098 | +0.0097 |
| 0.4 | +0.0315 | +0.0302 | +0.0289 | +0.0277 | +0.0265 | +0.0253 | +0.0242 | +0.0230 | +0.0220 | +0.0201 | +0.0203 |
| 0.5 | +0.0365 | +0.0350 | +0.0336 | +0.0322 | +0.0308 | +0.0295 | +0.0282 | +0.0270 | +0.0257 | +0.0255 | +0.0253 |

| D | 0.253 | 0.254 | 0.255 | 0.260 | 0.265 | 0.270 | 0.275 | 0.280 | 0.285 | 0.290 | 0.295 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | -0.0993 | -0.0933 | -0.0883 | -0.0710 | -0.0588 | -0.0516 | -0.0463 | -0.0412 | -0.0368 | -0.0331 | -0.0300 |
| 0.1 | -0.0471 | -0.0461 | -0.0452 | -0.0411 | -0.0374 | -0.0339 | -0.0309 | -0.0283 | -0.0259 | -0.0238 | -0.0218 |
| 0.2 | -0.0114 | -0.0113 | -0.0112 | -0.0106 | -0.0099 | -0.0093 | -0.0088 | -0.0082 | -0.0078 | -0.0072 | -0.0068 |
| 0.3 | +0.0096 | +0.0095 | +0.0094 | +0.0089 | +0.0084 | +0.0080 | +0.0076 | +0.0071 | +0.0066 | +0.0063 | +0.0059 |
| 0.4 | +0.0205 | +0.0207 | +0.0209 | +0.0199 | +0.0189 | +0.0180 | +0.0170 | +0.0162 | +0.0153 | +0.0145 | +0.0137 |
| 0.5 | +0.0251 | +0.0248 | +0.0245 | +0.0234 | +0.0223 | +0.0212 | +0.0201 | +0.0191 | +0.0181 | +0.0170 | +0.0160 |

TABLE A-1(Continued) TABULAR VALUES OF THE CORRECTION FACTOR δ

| P ↓ \ M → | 0.300 | 0.305 | 0.310 | 0.315 | 0.320 | 0.325 | 0.330 | 0.335 | 0.340 | 0.345 | 0.350 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | -0.0272 | -0.0247 | -0.0225 | -0.0206 | -0.0189 | -0.0173 | -0.0159 | -0.0146 | -0.0134 | -0.0123 | -0.0113 |
| 0.1 | -0.0200 | -0.0183 | -0.0168 | -0.0155 | -0.0142 | -0.0130 | -0.0120 | -0.0111 | -0.0102 | -0.0095 | -0.0089 |
| 0.2 | -0.0063 | -0.0059 | -0.0055 | -0.0051 | -0.0048 | -0.0045 | -0.0041 | -0.0038 | -0.0036 | -0.0033 | -0.0031 |
| 0.3 | +0.0056 | +0.0052 | +0.0049 | +0.0046 | +0.0043 | +0.0041 | +0.0038 | +0.0036 | +0.0033 | +0.0031 | +0.0029 |
| 0.4 | +0.0130 | +0.0122 | +0.0116 | +0.0109 | +0.0103 | +0.0097 | +0.0091 | +0.0086 | +0.0080 | +0.0075 | +0.0071 |
| 0.5 | +0.0152 | +0.0147 | +0.0140 | +0.0131 | +0.0123 | +0.0116 | +0.0110 | +0.0103 | +0.0097 | +0.0092 | +0.0086 |

| P | 0.355 | 0.360 | 0.365 | 0.370 | 0.375 | 0.380 | 0.385 | 0.390 | 0.395 | 0.400 | 0.41 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | -0.0104 | -0.0097 | -0.0089 | -0.0082 | -0.0076 | -0.0070 | -0.0064 | -0.0060 | -0.0055 | -0.0051 | -0.0043 |
| 0.1 | -0.0082 | -0.0075 | -0.0070 | -0.0065 | -0.0060 | -0.0055 | -0.0051 | -0.0047 | -0.0044 | -0.0040 | -0.0035 |
| 0.2 | -0.0029 | -0.0027 | -0.0025 | -0.0023 | -0.0021 | -0.0019 | -0.0018 | -0.0017 | -0.0016 | -0.0015 | -0.0013 |
| 0.3 | +0.0027 | +0.0025 | +0.0024 | +0.0022 | +0.0021 | +0.0019 | +0.0018 | +0.0017 | +0.0016 | +0.0015 | +0.0013 |
| 0.4 | +0.0066 | +0.0062 | +0.0058 | +0.0055 | +0.0051 | +0.0048 | +0.0044 | +0.0041 | +0.0039 | +0.0036 | +0.0031 |
| 0.5 | +0.0080 | +0.0075 | +0.0071 | +0.0066 | +0.0062 | +0.0058 | +0.0054 | +0.0050 | +0.0047 | +0.0047 | +0.0039 |

| P | 0.42 | 0.43 | 0.44 | 0.45 | 0.46 | 0.47 | 0.48 | 0.49 | 0.50 | 0.51 | 0.52 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | -0.0035 | -0.0031 | -0.0026 | -0.0023 | -0.0019 | -0.0017 | -0.0014 | -0.0012 | -0.0010 | -0.0009 | -0.0008 |
| 0.1 | -0.0030 | -0.0025 | -0.0022 | -0.0019 | -0.0016 | -0.0014 | -0.0011 | -0.0010 | -0.0008 | -0.0007 | -0.0006 |
| 0.2 | -0.0011 | -0.0010 | -0.0008 | -0.0007 | -0.0006 | -0.0005 | -0.0004 | -0.0004 | -0.0003 | -0.0003 | -0.0002 |
| 0.3 | +0.0012 | +0.0010 | +0.0008 | +0.0007 | +0.0006 | +0.0005 | +0.0004 | +0.0004 | +0.0003 | +0.0002 | +0.0002 |
| 0.4 | +0.0027 | +0.0023 | +0.0020 | +0.0018 | +0.0015 | +0.0013 | +0.0011 | +0.0009 | +0.0008 | +0.0007 | +0.0006 |
| 0.5 | +0.0035 | +0.0029 | +0.0025 | +0.0021 | +0.0018 | +0.0015 | +0.0013 | +0.0011 | +0.0010 | +0.0009 | +0.0007 |

| P | 0.53 | 0.54 | 0.55 |
|---|---|---|---|
| 0.0 | -0.0007 | -0.0006 | -0.0005 |
| 0.1 | -0.0005 | -0.0004 | -0.0003 |
| 0.2 | -0.0002 | -0.0001 | -0.0001 |
| 0.3 | +0.0002 | +0.0001 | +0.0001 |
| 0.4 | +0.0005 | +0.0005 | +0.0004 |
| 0.5 | +0.0006 | +0.0006 | +0.0005 |

NOTE: Values which are indeterminate are indicated by -.

TABLE A-2, TABULAR VALUES OF G

| D/E → ↓ | 0.100 | 0.105 | 0.110 | 0.115 | 0.120 | 0.125 | 0.130 | 0.135 | 0.140 | 0.145 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 2.000 | 2.000 | 2.000 | 2.000 | 1.999 | 1.999 | 1.999 | 1.998 | 1.998 | 1.997 |
| 0.1 | 2.255 | 2.230 | 2.209 | 2.191 | 2.174 | 2.160 | 2.147 | 2.135 | 2.124 | 2.114 |
| 0.2 | 3.082 | 2.973 | 2.879 | 2.747 | 2.726 | 2.663 | 2.608 | 2.559 | 2.514 | 2.474 |
| 0.3 | 4.662 | 4.364 | 4.112 | 3.897 | 3.711 | 3.550 | 3.408 | 3.283 | 3.172 | 3.073 |
| 0.4 | 7.064 | 6.409 | 5.868 | 5.416 | 5.034 | 4.708 | 4.427 | 4.184 | 3.972 | 3.786 |
| 0.5 | 8.701 | 7.746 | 6.972 | 6.338 | 5.811 | 5.368 | 4.993 | 4.672 | 4.394 | 4.154 |

| D/E | 0.150 | 0.155 | 0.160 | 0.165 | 0.170 | 0.180 | 0.190 | 0.200 | 0.210 | 0.220 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.996 | 1.995 | 1.994 | 1.993 | 1.992 | 1.989 | 1.985 | 1.980 | 1.975 | 1.970 |
| 0.1 | 2.105 | 2.097 | 2.089 | 2.081 | 2.074 | 2.061 | 2.048 | 2.036 | 2.024 | 2.012 |
| 0.2 | 2.438 | 2.404 | 2.374 | 2.346 | 2.320 | 2.293 | 2.232 | 2.195 | 2.162 | 2.133 |
| 0.3 | 2.984 | 2.904 | 2.831 | 2.765 | 2.705 | 2.599 | 2.509 | 2.431 | 2.364 | 2.305 |
| 0.4 | 3.621 | 3.475 | 3.344 | 3.227 | 3.122 | 2.941 | 2.791 | 2.666 | 2.560 | 2.470 |
| 0.5 | 3.943 | 3.758 | 3.595 | 3.449 | 3.320 | 3.099 | 2.918 | 2.769 | 2.645 | 2.540 |

| D/E | 0.230 | 0.240 | 0.250 | 0.260 | 0.270 | 0.280 | 0.290 | 0.300 | 0.310 | 0.320 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.963 | 1.957 | 1.949 | 1.942 | 1.934 | 1.925 | 1.917 | 1.908 | 1.899 | 1.890 |
| 0.1 | 2.001 | 1.989 | 1.978 | 1.967 | 1.956 | 1.944 | 1.933 | 1.923 | 1.912 | 1.901 |
| 0.2 | 2.106 | 2.081 | 2.057 | 2.036 | 2.016 | 1.997 | 1.979 | 1.962 | 1.946 | 1.931 |
| 0.3 | 2.254 | 2.208 | 2.167 | 2.130 | 2.097 | 2.066 | 2.039 | 2.014 | 1.991 | 1.970 |
| 0.4 | 2.392 | 2.324 | 2.265 | 2.214 | 2.168 | 2.127 | 2.091 | 2.058 | 2.029 | 2.002 |
| 0.5 | 2.450 | 2.373 | 2.306 | 2.248 | 2.198 | 2.152 | 2.112 | 2.076 | 2.044 | 2.015 |

| D/E | 0.330 | 0.340 | 0.350 | 0.360 | 0.370 | 0.380 | 0.390 | 0.400 | 0.450 | 0.500 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.881 | 1.873 | 1.864 | 1.855 | 1.846 | 1.838 | 1.829 | 1.821 | 1.783 | 1.750 |
| 0.1 | 1.891 | 1.881 | 1.871 | 1.861 | 1.852 | 1.843 | 1.834 | 1.825 | 1.785 | 1.751 |
| 0.2 | 1.917 | 1.903 | 1.890 | 1.878 | 1.866 | 1.855 | 1.845 | 1.834 | 1.789 | 1.753 |
| 0.3 | 1.950 | 1.932 | 1.915 | 1.900 | 1.885 | 1.871 | 1.858 | 1.846 | 1.795 | 1.756 |
| 0.4 | 1.970 | 1.956 | 1.936 | 1.917 | 1.900 | 1.885 | 1.970 | 1.856 | 1.800 | 1.758 |
| 0.5 | 1.989 | 1.966 | 1.944 | 1.924 | 1.906 | 1.890 | 1.874 | 1.860 | 1.802 | 1.759 |

| E | G | E | G | E | G | E | G |
|---|---|---|---|---|---|---|---|
| 0.550 | 1.723 | 1.00 | 1.585 | 3.00 | 1.472 | 6.00 | 1.443 |
| 0.600 | 1.697 | 1.25 | 1.551 | 3.25 | 1.468 | 6.50 | 1.441 |
| 0.650 | 1.675 | 1.50 | 1.529 | 3.50 | 1.464 | 7.00 | 1.439 |
| 0.700 | 1.657 | 1.75 | 1.513 | 3.75 | 1.461 | 7.50 | 1.438 |
| 0.750 | 1.641 | 2.00 | 1.501 | 4.00 | 1.458 | 8.00 | 1.436 |
| 0.800 | 1.627 | 2.25 | 1.491 | 4.50 | 1.453 | 8.50 | 1.435 |
| 0.850 | 1.615 | 2.50 | 1.484 | 5.00 | 1.449 | 9.00 | 1.433 |
| 0.900 | 1.604 | 2.75 | 1.477 | 5.50 | 1.446 | 10.00 | 1.432 |

**NOTE:** These values of G depend upon E only.

TABLE A-3 TABULAR VALUES OF H

| p/E→ | 0.100 | 0.105 | 0.110 | 0.115 | 0.120 | 0.125 | 0.130 | 0.135 | 0.140 | 0.145 | 0.150 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 14.842 | 12.283 | 10.364 | 8.892 | 7.742 | 6.827 | 6.088 | 5.484 | 4.983 | 4.564 | 4.210 |
| 0.1 | 2.212 | 2.276 | 2.333 | 2.381 | 2.419 | 2.446 | 2.462 | 2.467 | 2.462 | 2.447 | 2.425 |
| 0.2 | 1.510 | 1.518 | 1.527 | 1.536 | 1.545 | 1.553 | 1.559 | 1.565 | 1.569 | 1.570 | 1.572 |
| 0.3 | 1.493 | 1.445 | 1.424 | 1.398 | 1.375 | 1.356 | 1.340 | 1.326 | 1.315 | 1.305 | 1.296 |
| 0.4 | 1.646 | 1.559 | 1.487 | 1.427 | 1.377 | 1.334 | 1.299 | 1.269 | 1.243 | 1.222 | 1.204 |
| 0.5 | 1.740 | 1.627 | 1.534 | 1.458 | 1.395 | 1.342 | 1.298 | 1.261 | 1.230 | 1.204 | 1.183 |
| | 0.155 | 0.160 | 0.165 | 0.170 | 0.180 | 0.190 | 0.200 | 0.210 | 0.220 | 0.230 | 0.240 |
| 0.0 | 3.908 | 3.649 | 3.424 | 3.228 | 2.906 | 2.654 | 2.453 | 2.290 | 2.157 | 2.048 | 1.956 |
| 0.1 | 2.396 | 2.361 | 2.324 | 2.283 | 2.198 | 2.114 | 2.034 | 1.960 | 1.893 | 1.834 | 1.781 |
| 0.2 | 1.572 | 1.571 | 1.568 | 1.565 | 1.556 | 1.546 | 1.534 | 1.521 | 1.509 | 1.497 | 1.486 |
| 0.3 | 1.289 | 1.283 | 1.278 | 1.274 | 1.268 | 1.265 | 1.264 | 1.265 | 1.268 | 1.271 | 1.276 |
| 0.4 | 1.189 | 1.177 | 1.167 | 1.59 | 1.148 | 1.142 | 1.141 | 1.143 | 1.148 | 1.155 | 1.164 |
| 0.5 | 1.165 | 1.150 | 1.138 | 1.128 | 1.114 | 1.107 | 1.105 | 1.107 | 1.112 | 1.120 | 1.129 |
| | 0.250 | 0.260 | 0.270 | 0.280 | 0.290 | 0.300 | 0.310 | 0.320 | 0.330 | 0.340 | 0.350 |
| 0.0 | 1.880 | 1.815 | 1.761 | 1.714 | 1.674 | 1.641 | 1.612 | 1.587 | 1.567 | 1.549 | 1.534 |
| 0.1 | 1.734 | 1.693 | 1.658 | 1.627 | 1.600 | 1.577 | 1.557 | 1.540 | 1.525 | 1.513 | 1.502 |
| 0.2 | 1.476 | 1.467 | 1.459 | 1.452 | 1.446 | 1.441 | 1.437 | 1.434 | 1.431 | 1.430 | 1.429 |
| 0.3 | 1.281 | 1.287 | 1.294 | 1.300 | 1.307 | 1.315 | 1.322 | 1.329 | 1.337 | 1.344 | 1.351 |
| 0.4 | 1.174 | 1.185 | 1.97 | 1.209 | 1.222 | 1.235 | 1.248 | 1.260 | 1.273 | 1.285 | 1.298 |
| 0.5 | 1.140 | 1.153 | 1.166 | 1.180 | 1.194 | 1.208 | 1.222 | 1.237 | 1.251 | 1.265 | 1.279 |
| | 0.360 | 0.370 | 0.380 | 0.390 | 0.400 | 0.450 | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 |
| 0.0 | 1.521 | 1.511 | 1.502 | 1.495 | 1.490 | 1.478 | 1.485 | 1.502 | 1.524 | 1.550 | 1.577 |
| 0.1 | 1.494 | 1.487 | 1.481 | 1.477 | 1.473 | 1.470 | 1.480 | 1.499 | 1.523 | 1.549 | 1.576 |
| 0.2 | 1.428 | 1.429 | 1.430 | 1.431 | 1.433 | 1.447 | 1.468 | 1.492 | 1.519 | 1.547 | 1.575 |
| 0.3 | 1.359 | 1.366 | 1.393 | 1.380 | 1.387 | 1.421 | 1.453 | 1.484 | 1.515 | 1.546 | 1.574 |
| 0.4 | 1.309 | 1.321 | 1.332 | 1.343 | 1.353 | 1.401 | 1.442 | 1.478 | 1.511 | 1.543 | 1.573 |
| 0.5 | 1.292 | 1.305 | 1.317 | 1.329 | 1.341 | 1.393 | 1.437 | 1.475 | 1.510 | 1.542 | 1.573 |

| E | H | E | H | E | H | E | H | E | H |
|---|---|---|---|---|---|---|---|---|---|
| 0.75 | 1.627 | 2.00 | 2.239 | 4.00 | 3.001 | 6.00 | 3.606 | 8.00 | 4.123 |
| 0.80 | 1.632 | 2.25 | 2.348 | 4.25 | 3.083 | 6.25 | 3.674 | 8.25 | 4.183 |
| 0.85 | 1.661 | 2.50 | 2.452 | 4.50 | 3.163 | 6.50 | 3.742 | 8.50 | 4.243 |
| 0.90 | 1.689 | 2.75 | 2.551 | 4.75 | 3.241 | 6.75 | 3.808 | 8.75 | 4.301 |
| 1.00 | 1.745 | 3.00 | 2.647 | 5.00 | 3.317 | 7.00 | 3.873 | 9.00 | 4.359 |
| 1.25 | 1.879 | 3.25 | 2.740 | 5.25 | 3.392 | 7.25 | 3.937 | 9.25 | 4.417 |
| 1.50 | 2.006 | 3.50 | 2.829 | 5.50 | 3.464 | 7.50 | 4.000 | 9.50 | 4.475 |
| 1.75 | 2.126 | 3.75 | 2.916 | 5.75 | 3.536 | 7.75 | 4.062 | 9.75 | 4.537 |

TABLE A-4 COMPILATION OF EQUATIONS NEEDED TO COMPUTE THE LOGIT RESPONSE LEVEL AND CONFIDENCE LIMITS

1. To compute response stimulus level, $x_p$; probability of response is p.

$$x_p = \bar{x} + g \cdot \ln\left(\frac{p}{1-p}\right) = \bar{x} + g \cdot L(x)$$

$$L(x) = \ln\left(\frac{p}{1-p}\right)$$

2. To compute the standard deviation of $x_p$.

$$s_p = \sqrt{(s_m)^2 + [L(x_p)]^2 (s_g)^2}$$

3. To approximate the value of Student's t.

$$t = R + \frac{1}{Sf-V}$$

| Single-Sided Confidence Level, P (%) | R | S | V | Two-Sided Confidence Level P (%) | Usable for f ≥ |
|---|---|---|---|---|---|
| 90 | 1.2820 | 1.2229 | 1.1345 | 80 | 9 |
| 95 | 1.6450 | 0.6602 | 0.6279 | 90 | 7 |
| 97.5 | 1.9600 | 0.4210 | 0.4774 | 95 | 6 |
| 99 | 2.3260 | 0.26687 | 0.37980 | 98 | 7 |
| 99.5 | 2.5760 | 0.20276 | 0.33618 | 99 | 7 |

4. To compute two-sided interval about $x_p$ at a confidence of P%:

$$(L,U) = [(x_p - t \cdot s_p), (x_p + t \cdot s_p)];$$

At the 50% point:

$$(L,U) = [(\bar{x} - t \cdot s_m), (\bar{x} + t \cdot s_m)].$$

5. To compute the lower single-sided limit at a confidence of P%:

$$(L) = x_p - t \cdot s_p \; .$$

6. To compute the upper single-sided limit at a confidence of P%:

$$(U) = x_p + t \cdot s_p \; .$$

# APPENDIX B

## DERIVATION OF VARIOUS MOMENTS OF THE LOGISTIC DISTRIBUTION

1. Ordinarily the logistic distribution is expressed in the cumulative form:

$$L(x) = \ln \frac{p(x)}{1-p(x)} = \frac{x-\mu}{\gamma}\ , \tag{B-1}$$

where p is the cumulative probability at a stimulus level of x. The p.d.f. (probability density function) is defined as the derivative of the cumulative probability:

$$p(x) = \frac{dp(x)}{dx} = \frac{d[p(x)]}{dx} \tag{B-2}$$

Equation (B-1) can be solved for p to give

$$p(x) = \frac{e^{y}}{1+e^{y}} = \frac{1}{1+e^{-y}}\ , \text{ where} \tag{B-3}$$

$$y = \frac{x-\mu}{\gamma}\ .$$

By letting $u = 1 + e^{-y}$, the equations can be made less awkward to write

$$p(x) = \frac{1}{u}\ . \tag{B-4}$$

The p.d.f., then, becomes

$$\frac{d[p(x)]}{dx} = \frac{d[p(x)]}{du} \cdot \frac{du}{dy} \cdot \frac{dy}{dx} \tag{B-5}$$

and the individual derivatives can be seen to be

$$\frac{dy}{dx} = \frac{1}{\gamma}\ , \tag{B-6}$$

$$\frac{du}{dy} = -e^{-y}, \text{ and} \tag{B-7}$$

$$\frac{d[p(x)]}{du} = -\frac{1}{u^2}\ . \tag{B-8}$$

Thus,

$$\frac{d[p(x)]}{dx} = \text{p.d.f.} = \left(\frac{1}{u^2}\right)\left(e^{-y}\right)\left(\frac{1}{\gamma}\right) \tag{B-9}$$

$$= \frac{1}{\gamma}\left(\frac{e^{-y}}{1+2e^{-y}+e^{-2y}}\right) \qquad \text{(B-10)}$$

$$= \frac{1}{\gamma}\left(\frac{1}{2+2\left(\frac{e^{y}+e^{-y}}{2}\right)}\right) \qquad \text{(B-11)}$$

$$= \frac{1}{2\gamma}\left(\frac{1}{1+\cosh(y)}\right) \qquad \text{(B-12)}$$

$$= \frac{1}{2\gamma}\left(\frac{1}{1+\cosh\left(\frac{x-\mu}{\gamma}\right)}\right) \qquad \text{(B-13)}$$

2. The integral of the p.d.f., between the limits of $-\infty$ and $+\infty$, should be equal to 1 for a well-behaved distribution function:

$$A_1 = \frac{1}{2\gamma}\int_{-\infty}^{+\infty}\frac{dx}{1+\cosh\left(\frac{x-\mu}{\gamma}\right)} \qquad \text{(B-14)}$$

since

$$y = \frac{x-u}{\gamma}$$

$$dy = \frac{dx}{\gamma}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{dy}{1+\cosh y} \qquad \text{(B-15)}$$

$$= \int_{-\infty}^{+\infty}\frac{dy}{2+\gamma\left(\frac{e^{y}-e^{-y}}{2}\right)} \qquad \text{(B-16)}$$

let $w = e^{y}$; $dw = e^{y}dy$

$$= \int_{0}^{+\infty}\frac{dw}{(w+1)^2} \qquad \text{(B-17)}$$

| Limits | | |
|---|---|---|
| x | y | w |
| $+\infty$ | $+\infty$ | $+\infty$ |
| $-\infty$ | $-\infty$ | 0 |

$$= -\left(\frac{1}{w+1}\right)\Big]_{0}^{\infty}$$

$$= -0 - (-1)$$

$$= 1 \ldots \text{Q.E.D.}$$

3. By definition, the mean should be the first moment about the origin:

$$M_1 = \text{mean} = \int_{-\infty}^{+\infty} (\text{p.d.f.})x\,dx \;. \tag{B-18}$$

In our notation we should be able to show that $M_1$ equals $\mu$. Since $x = \mu + \gamma y$ equation (B-18) becomes

$$M_1 = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(\mu+\gamma y)\,dy}{1+\cosh y} \tag{B-19}$$

$$= \frac{\mu}{2}\int_{-\infty}^{+\infty} \frac{dy}{1+\cosh y} + \frac{\mu}{2}\int_{-\infty}^{+\infty} \frac{y\,dy}{1+\cosh y} \;. \tag{B-20}$$

But the second term is an odd function. The integral of an odd function between limits which are symmetric about zero, is itself zero. The first term of the above equation is the same as equation (B-15), multiplied by $\mu$. Hence

$$M_1 = \mu A_1 = \mu \text{ Q.E.D.}$$

4. By definition, the variance should be the second moment about the mean:

$$M_2 = \sigma^2 = \int_{-\infty}^{+\infty} (\text{p.d.f.})(x - \text{mean})^2\,dx \tag{B-21}$$

$$= \frac{1}{2\gamma}\int_{-\infty}^{+\infty} \frac{(x-\mu)^2\,dx}{1+\cosh\left(\frac{x-\mu}{\gamma}\right)} \tag{B-22}$$

Again, since $x = \mu + \gamma y$ the preceding equation becomes

$$\sigma^2 = \frac{\gamma^2}{2}\int_{-\infty}^{\infty} \frac{y^2\,dy}{1+\cosh y} \;, \text{ and} \tag{B-23}$$

$$= \gamma^2 \int_{0}^{\infty} \frac{y^2\,dy}{1+\cosh y} \tag{B-24}$$

but,

$$\frac{1}{1+\cosh y} = \frac{2e^y}{(1+e^y)^2}, \tag{B-25}$$

and thus

$$\sigma^2 = 2\gamma^2 \int_0^\infty \frac{y^2 e^y dy}{(e^y+1)^2}. \tag{B-26}$$

By breaking (B-26) into $y^2$ and $\frac{e^y dy}{(e^y+1)^2}$ and integrating by parts the expression becomes

$$\sigma^2 = 2\gamma^2 \left[\frac{-y^2}{e^y+1} - \int - \frac{2y dy}{e^y+1}\right]_0^{+\infty}.$$

But the term, $-\frac{y^2}{e^y+1}$, is zero at $y = o$ and becomes zero as $y \to \infty$; thus

$$\sigma^2 = 4\gamma^2 \int_0^\infty \frac{y dy}{e^y+1}.$$

If now y is transformed by $y = \ln u$, then $dy = \frac{du}{u}$ and

| Limits | |
|---|---|
| y | u |
| $\infty$ | $\infty$ |
| o | 1 |

$$\sigma^2 = 4\gamma^2 \int_1^\infty \frac{\ln u \, du}{u(1+u)}.$$

let $$v = \frac{1}{u}.$$

| Limits | |
|---|---|
| u | v |
| $\infty$ | o |
| 1 | 1 |

$$\sigma^2 = 4\gamma^2 \int_1^0 \frac{(-\ln v)(-\frac{dv}{v^2})}{\frac{1}{v}(1+\frac{1}{v})}$$

$$= 4\gamma^2 \int_0^1 - \frac{\ln v \, dv}{1+v} .$$

Breaking this into ln v and $\frac{dv}{v}$, and integrating by parts, yields

$$\sigma^2 = 4\gamma^2 \left[ -(\ln v) \ln (1+v) + \int \frac{\ln(1+v)}{v} dv \right]_0^1 .$$

The term $(\ln v)\ln(1+v)$ is zero when $v = 1$ but becomes indeterminate at $v = o$. By rewriting the term in the form

$$\frac{\ln v}{\dfrac{1}{\ln(1+v)}}$$

and applying l'Hospital's rule we find that the term also vanishes at $v = o$. Hence

$$\sigma^2 = 4\gamma^2 \int_0^1 \frac{\ln(1+v) dv}{v} .$$

Now, a well known series exists:

$$\ln(1+v) = \frac{v}{1} - \frac{v^2}{2} + \frac{v^3}{3} \cdots\cdots\cdots .$$

The integral between the limits of zero and one gives

$$\int_0^1 \frac{\ln(1+v) dv}{v} = v - \frac{v^2}{2^2} + \frac{v^3}{3^2} \cdots\cdots\cdots \Big]_0^1 = \frac{\pi^2}{12} .$$

Therefore

$$\sigma^2 = 4\gamma^2 \frac{\pi^2}{12} , \text{ and } \sigma = \frac{\pi}{\sqrt{3}} \gamma .$$

5. The preceeding derivation of the variance of the logistic distribution was performed by Dr. A. H. Van Tuyl of this Laboratory. Independent of his effort were two other pieces of work. One of these was a derivation of the variance (by Mr. Blum) following a markedly different chain of logic. The other was an experimental verification using a high-speed Monte Carlo simulation. A description of the second derivation and of the Monte Carlo experiment follow, in that order.

6. Derivation of Standard Deviation of Logistic Distribution

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\,dx \qquad p(x) = \frac{e^{-\frac{(x-\mu)}{\gamma}}}{\gamma\left(1+e^{-\frac{(x-\mu)}{\gamma}}\right)^2}$$

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{(x-\mu)^2 e^{-\frac{(x-\mu)}{\gamma}}\,dx}{\gamma\left(1+e^{-\frac{(x-\mu)}{\gamma}}\right)^2} \qquad \text{let } y = \frac{x-\mu}{\gamma}, \quad \frac{dx}{\gamma} = dy$$

$$\sigma^2 = \gamma^2 \int_{-\infty}^{\infty} \frac{y^2 e^{-y}\,dy}{(1+e^{-y})^2}$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial K^2} \left( \frac{e^{-yK}}{[1+e^{-y}]^2} \right) dy$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial K^2}\, e^{-y(K-1)} \left( \frac{e^{-y}}{[1+e^{-y}]^2} \right) dy$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} \int_{-\infty}^{\infty} \frac{[e^{-y}]^{(K-1)} e^{-y}\,dy}{(1+e^{-y})^2}$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} \int_0^\infty \frac{u^{K-1} du}{(1+u)^2} \qquad \text{let } u = e^{-y},\ du = -e^{-y} dy$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} B(K, 2-K) \qquad B(m,n) = \int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} \left[\frac{\Gamma(K)\Gamma(2-K)}{\Gamma(2)}\right] \qquad \Gamma(n) = (n-1)!$$

$$\Gamma(X+1) = X\Gamma(X)$$

$$\Gamma(K)\Gamma(1-K) = \frac{\pi}{\sin \pi K}$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} \left[\Gamma(K) \cdot \Gamma(1-K) \cdot (1-K)\right]$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial^2}{\partial K^2} \left[\frac{\pi(1-K)}{\sin \pi K}\right]$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \frac{\partial}{\partial K} \left[\frac{-\pi}{\sin \pi K} + \frac{\pi^2 (K-1) \cos \pi K}{\sin^2 \pi K}\right]$$

$$\sigma^2 = \gamma^2 \lim_{K \to 1} \left[\frac{\pi^2 \cos \pi K}{\sin^2 \pi K} + \frac{\sin \pi K (\pi^2 \cos \pi K - (K-1)\pi^3 \sin \pi K) - 2(K-1)\pi^3 \cos^2 \pi K}{\sin^3 \pi K}\right]$$

$$\sigma^2 = \gamma^2 \pi^2 \lim_{K \to 1} \left[\frac{\sin 2\pi K + \pi(1-K)(1+\cos^2 \pi K)}{\sin^3 \pi K}\right]$$

$$\sigma^2 = \gamma^2 \pi^2 \lim_{K \to 1} \left[\frac{2\cos(2\pi K) - 1 - \cos^2(\pi K) + \pi(K-1)\sin(2\pi K)}{3 \sin^2 \pi K \cos \pi K}\right] \text{ by L'Hospital's rule}$$

$$\sigma^2 = Y^2\pi^2 \cdot \frac{2}{3} \lim_{K \to 1} \left[\frac{\sin 2\pi K + \pi(1-K)\cos 2\pi K}{\sin 2\pi K}\right]$$

$$\sigma^2 = Y^2\pi^2 \cdot \frac{2}{3}\left(1 - \frac{1}{2}\right) = Y^2 \frac{\pi^2}{3} = 3.2898\ Y^2 \quad .$$

Reference: Apostol, Mathematical Analysis. [Theorem 14-24:]
The order of differentiation and improper integration may be exchanged under the following conditions:

(1) $\int_a^b \frac{e^{-Y}dy}{(1+e^{-Y})^2}$ exists for all a,b in $(-\infty,\infty)$;

(2) $\int_{-\infty}^{\infty} \frac{e^{-Y}K dy}{(1+e^{-Y})^2}$ converges pointwise for all K in [c,d] for some c,d;

(3) $\frac{\partial^2}{\partial K^2}\left[\frac{e^{-yK}}{(1+e^{-Y})^2}\right] = \frac{y^2 e^{-yK}}{(1+e^{-Y})^2}$ must be continuous on the strip $(-\infty,\infty)\times[c,d]$;

(4) $\int_{-\infty}^{\infty} \frac{\partial^2}{\partial K^2}\left[\frac{e^{-yK}}{(1+e^{-Y})^2}\right] dy$ must converge uniformly on (c,d).

(1) and (2) follow by limit-comparison with $e^{py}$ for appropriate values of p [theorem 14-4]; (3, follows from the continuity of the sum, product, and quotient of two continuous functions [theorem 4-10]; and (4) follows from a Cauchy condition [theorem 14-18]; if we choose any c, d satisfying $0<c<d<2$.

7. Monte Carlo studies on asymptotic convergence of the variance of the logistic distribution function.

We wished to verify the algorithm for producing numbers distributed randomly in the logistic distribution and at the same time demonstrate the nature of the effect of the sample size on the sampling error of the variance.

We assume that we know the critical levels of the devices being tested. Furthermore, we set the population parameters to convenient values... $\mu = 0$ and $\gamma = 1$. The logistic equation reduces to:

$$L(x) = \ln\left(\frac{p}{1-p}\right) = \frac{x-\mu}{\gamma} = x\ .$$

Using a high-speed computer it is possible to generate a sequence of numbers $p_1, p_2, p_3, p_4$.........which have the property of falling between the limits of zero and one and having a rectangular distribution. That is, if any particular value is chosen between zero and one that value will be the same, in the limit, as the decimal fraction of the numbers that will be generated which will fall between zero and that particular value. For instance if a value of .85 is chosen and if many hundreds of the random numbers are generated and classified according to whether they are larger or smaller than .85 we will find that 85 out of 100 on the average will fall in the lower classification.

The random rectangular distribution was converted to a random logistic distribution of values of x (with a $\mu$ of 0 and a $\gamma$ of 1) merely by letting

$$x_i = \ln\left(\frac{p_i}{1-p_i}\right)\ .$$

In the Monte Carlo experiment 100 trials were run on each of 9 sample sizes where the sample size, N, took values of 15, 20, 40, 60, 80, 100, 200, 500 and 1000. For each trial of $N_j$ samples a variance was computed using the equation

$$v = \frac{\Sigma x^2 - \frac{(\Sigma x)^2}{N_j}}{N_j - 1}\ .$$

When one hundred such variances had been obtained the average of these was computed as well as the 95% confidence band about the average. We also noted the smallest and largest value of v for each of the 9 different sample sizes. The observed values given in Table B-1 have been plotted in Figure B-1. To facilitate comparison with the theoretical value, $\pi^2/3$, the observed values normalized by dividing by $\pi^2/3$, which is 3.28987, are also tabulated.

TABLE B-1

ASYMPTOTIC CONVERGENCE OF THE LOGISTIC DISTRIBUTION FUNCTION

| N Sample Size | Observed Minimum | Lower Limit of 95% Confidence Band | Average Variance for 100 trials | Upper Limit of 95% Confidence Band | Observed Maximum | |
|---|---|---|---|---|---|---|
| 15 | 1.070 | 2.936 | 3.040 | 3.143 | 8.518 | Variance Values |
| 20 | 1.162 | 3.161 | 3.286 | 3.411 | 6.736 | |
| 40 | 1.766 | 3.156 | 3.260 | 3.363 | 7.732 | |
| 60 | 1.893 | 3.240 | 3.318 | 3.396 | 6.392 | |
| 80 | 2.139 | 3.288 | 3.351 | 3.414 | 5.099 | |
| 100 | 1.962 | 3.362 | 3.428 | 3.494 | 5.353 | |
| 200 | 2.362 | 3.263 | 3.300 | 3.338 | 4.161 | |
| 500 | 2.748 | 3.268 | 3.293 | 3.319 | 3.906 | |
| 1000 | 2.779 | 3.262 | 3.282 | 3.300 | 3.694 | |
| 15 | .325 | .862 | .924 | .986 | 2.589 | Variance Values Normalized to $\pi^2/3$ |
| 20 | .353 | .925 | .999 | 1.073 | 2.048 | |
| 40 | .537 | .929 | .991 | 1.052 | 2.350 | |
| 60 | .575 | .962 | 1.008 | 1.055 | 1.943 | |
| 80 | .650 | .981 | 1.019 | 1.056 | 1.550 | |
| 100 | .596 | 1.003 | 1.042 | 1.081 | 1.627 | |
| 200 | .718 | .981 | 1.003 | 1.025 | 1.265 | |
| 500 | .835 | .986 | 1.001 | 1.016 | 1.187 | |
| 1000 | .845 | .986 | 0.997 | 1.009 | 1.123 | |

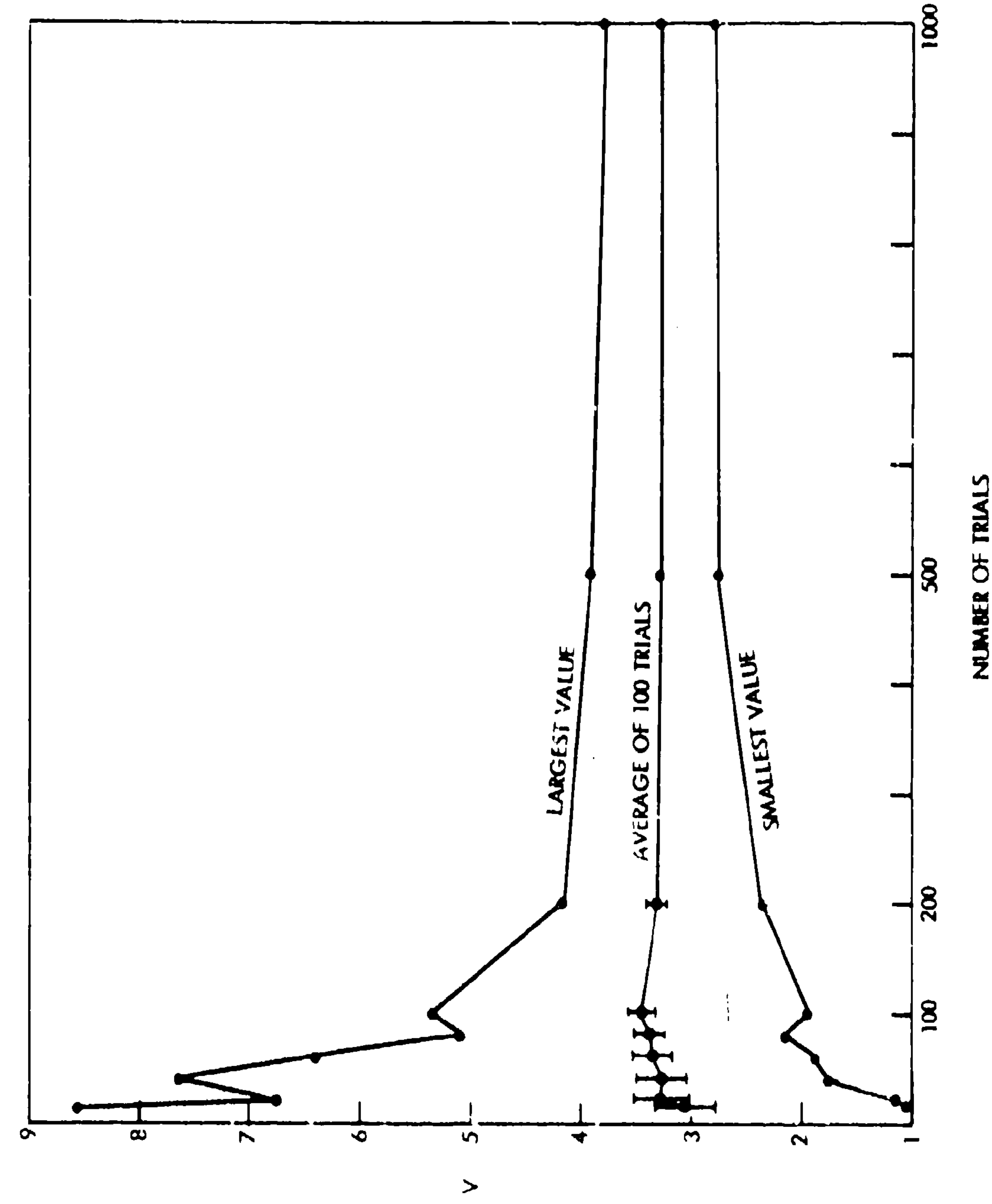


FIG. B-1 ASYMPTOTIC CONVERGENCE OF THE VARIANCE, V, OF THE LOGISTIC DISTRIBUTION FUNCTION

TABLE C-1

**MONTE CARLO RESULTS, SAMPLE SIZE 10, POPULATION MEAN 19.5**

| Rela. Step $d/\gamma$ | GAMMA Pop. $\gamma$ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09826 | 0.09800 | 19.49511 | 19.49800 | 0.05330 | 0.03077 | 1.2357 | 0.7134 | 1.0027 |
| 6.67 | 0.15 | 0.15304 | 0.14608 | 19.49702 | 19.51000 | 0.13165 | 0.07017 | 2.2667 | 1.2582 | 1.0476 |
| 4.00 | 0.25 | 0.23758 | 0.21159 | 19.49056 | 19.52500 | 0.19403 | 0.09910 | 2.1130 | 1.0792 | 1.1231 |
| 3.00 | 0.33 | 0.32035 | 0.25003 | 19.52385 | 19.53500 | 0.23736 | 0.11144 | 2.0885 | 0.9806 | 1.2812 |
| 2.00 | 0.50 | 0.51583 | 0.33573 | 19.47650 | 19.55350 | 0.43689 | 0.22381 | 2.9098 | 1.4907 | 1.5364 |
| 1.43 | 0.70 | 0.65514 | 0.35273 | 19.53856 | 19.59750 | 0.42559 | 0.38052 | 2.6980 | 2.4122 | 1.8574 |
| 1.18 | 0.85 | 0.81328 | 0.45129 | 19.42836 | 19.59633 | 0.56151 | 0.35111 | 2.7822 | 1.7397 | 1.8021 |
| 1.00 | 1.00 | 0.99700 | 0.43588 | 19.38870 | 19.50683 | 0.67316 | 0.37206 | 3.4533 | 1.9087 | 2.2873 |
| 0.67 | 1.50 | 1.31451 | 0.63267 | 19.48020 | 19.56566 | 0.92517 | 0.44932 | 3.2698 | 1.5880 | 2.0777 |
| 0.50 | 2.00 | 1.89733 | 0.76910 | 19.59100 | 19.65316 | 1.08617 | 0.81273 | 3.1579 | 2.3629 | 2.4669 |
| 0.40 | 2.50 | 2.44642 | 0.74679 | 19.53989 | 19.76233 | 1.26309 | 0.74596 | 3.7820 | 2.2336 | 3.2759 |
| 0.33 | 3.00 | 2.74672 | 0.67226 | 19.67070 | 19.58766 | 1.20888 | 0.48432 | 4.0210 | 1.6109 | 4.0858 |
| 0.25 | 4.00 | 3.94066 | 0.82584 | 20.03023 | 19.89550 | 1.43886 | 0.72124 | 3.8959 | 1.9529 | 4.7717 |
| 0.20 | 5.00 | 4.79313 | 0.99453 | 19.81754 | 19.97200 | 1.74861 | 1.19603 | 3.9315 | 2.6891 | 4.8195 |
| 0.13 | 7.50 | 7.25214 | 1.01271 | 19.07776 | 19.48750 | 1.59640 | 1.12184 | 3.5248 | 2.4770 | 7.1611 |
| 0.10 | 10.00 | 9.68170 | 0.92879 | 19.51327 | 19.91816 | 1.62438 | 1.07695 | 3.9107 | 2.5927 | 10.4240 |
| 0.05 | 20.00 | 18.65962 | 1.05679 | 17.84260 | 19.78183 | 1.86641 | 1.16997 | 3.9491 | 2.4755 | 17.6568 |
| 0.02 | 50.00 | 47.53511 | 0.80660 | 13.82722 | 19.59550 | 2.83956 | 1.13535 | 7.8718 | 3.1474 | 58.9324 |
| 0.01 | 100.00 | 94.71840 | 0.98269 | 13.93514 | 19.50117 | 1.91920 | 1.25812 | 4.3671 | 2.8628 | 96.3871 |

TABLE C-2

**MONTE CARLO RESULTS, SAMPLE SIZE 10, POPULATION MEAN 20.0**

| Rela. Step d/γ | GAMMA Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_p$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.10 | 0.09983 | | 20.00165 | 20.00200 | 0.23019 | | 2.6048 | | |
| 6.67 | 0.15 | 0.14756 | | 19.99174 | 19.96950 | 0.22213 | | 2.4800 | | |
| 4.00 | 0.25 | 0.24345 | | 20.01897 | 20.02400 | 0.28934 | | 3.0148 | | |
| 3.00 | 0.33 | 0.32118 | | 20.00088 | 20.03100 | 0.26335 | | 2.2100 | | |
| 2.00 | 0.50 | 0.47555 | 0.30260 | 20.01011 | 20.01100 | 0.35295 | 0.22305 | 2.6081 | 1.6482 | 1.5715 |
| 1.43 | 0.70 | 0.62745 | 0.40338 | 19.99699 | 20.00550 | 0.46670 | 0.31815 | 2.5871 | 1.7636 | 1.5555 |
| 1.18 | 0.85 | 0.78952 | 0.42085 | 19.95638 | 19.93016 | 0.59004 | 0.27739 | 3.1350 | 1.4738 | 1.8760 |
| 1.00 | 1.00 | 0.93549 | 0.46650 | 20.02954 | 20.07400 | 0.55694 | 0.27739 | 2.6696 | 1.5850 | 2.0053 |
| 0.67 | 1.50 | 1.43934 | 0.64950 | 19.90378 | 19.89883 | 0.97561 | 0.50303 | 3.3588 | 1.7318 | 2.2161 |
| 0.50 | 2.00 | 1.82667 | 0.63169 | 20.00347 | 19.96250 | 0.99947 | 0.45835 | 3.5380 | 1.6225 | 2.8918 |
| 0.40 | 2.50 | 2.39003 | 0.78361 | 19.79512 | 19.81550 | 1.19920 | 0.71001 | 3.4220 | 2.0261 | 3.0500 |
| 0.33 | 3.00 | 3.01784 | 0.76501 | 20.11989 | 20.14516 | 1.18449 | 0.65413 | 3.4622 | 1.9120 | 3.9449 |
| 0.25 | 4.00 | 3.77786 | 0.86030 | 20.21527 | 19.97866 | 1.34814 | 1.07553 | 3.5041 | 2.7955 | 4.3913 |
| 0.20 | 5.00 | 4.67088 | 0.87271 | 20.01062 | 20.02500 | 1.45484 | 0.69221 | 3.7276 | 1.7736 | 5.3522 |
| 0.13 | 7.50 | 7.31904 | 0.78676 | 20.20552 | 20.04233 | 1.75623 | 0.78221 | 4.9914 | 2.2231 | 9.3028 |
| 0.10 | 10.00 | 9.56486 | 0.74183 | 20.80147 | 20.10583 | 1.72748 | 1.03409 | 5.2071 | 3.1170 | 12.8937 |
| 0.05 | 20.00 | 19.32119 | 1.10902 | 20.65520 | 19.96583 | 1.90971 | 1.27321 | 3.8505 | 2.5671 | 17.4219 |
| 0.02 | 50.00 | 47.50245 | 0.99264 | 24.76285 | 19.98083 | 1.98675 | 1.32414 | 4.4754 | 2.9828 | 47.8545 |
| 0.01 | 100.00 | 95.55083 | 0.98665 | 17.70767 | 19.89716 | 1.92235 | 1.11790 | 4.3567 | 2.5335 | 96.8433 |

TABLE C-3

**MONTE CARLO RESULTS, SAMPLE SIZE 10, POPULATION MEAN 20.5**

| Rela. Step d/γ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. γ | Sample $g_s$ | BCTN $g_B$ | Sample $m_s$ | BCTN $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | $g_s/g_B$ |
| 10.00 | 0.1 | 0.09648 | 0.09465 | 20.51618 | 20.50450 | 0.05137 | 0.02953 | 1.1910 | 0.6846 | 1.0003 |
| 6.67 | 0.15 | 0.14049 | 0.12987 | 20.50407 | 20.49750 | 0.09343 | 0.06270 | 1.6087 | 1.0796 | 1.0818 |
| 4.00 | 0.25 | 0.23269 | 0.20534 | 20.48769 | 20.43900 | 0.21185 | 0.09752 | 2.3070 | 1.0620 | 1.1332 |
| 3.00 | 0.333 | 0.31138 | 0.25413 | 20.49393 | 20.44450 | 0.27017 | 0.11372 | 2.4564 | 1.0006 | 1.2253 |
| 2.00 | 0.5 | 0.48025 | 0.32310 | 20.48353 | 20.43600 | 0.33468 | 0.20692 | 2.3162 | 1.4320 | 1.4864 |
| 1.43 | 0.7 | 0.67816 | 0.37998 | 20.53566 | 20.44950 | 0.47570 | 0.25154 | 2.7994 | 1.4802 | 1.7847 |
| 1.18 | 0.85 | 0.83258 | 0.52733 | 20.54436 | 20.47050 | 0.56795 | 0.45139 | 2.4093 | 1.9140 | 1.5789 |
| 1.00 | 1.0 | 0.91987 | 0.45900 | 20.45805 | 20.32900 | 0.67704 | 0.36616 | 3.2982 | 1.7838 | 2.0041 |
| 0.67 | 1.5 | 1.41351 | 0.66311 | 20.37394 | 20.30983 | 0.84165 | 0.51629 | 1.331 | 1.7410 | 2.1316 |
| 0.50 | 2.0 | 1.93504 | 0.67832 | 20.30843 | 20.24350 | 1.10223 | 0.49060 | 1.274 | 1.6173 | 2.8527 |
| 0.40 | 2.5 | 2.51858 | 0.75393 | 20.59708 | 20.31650 | 1.21921 | 0.66012 | 1.082 | 1.9578 | 3.3406 |
| 0.33 | 3.0 | 2.85676 | 0.91189 | 20.50440 | 20.17483 | 1.08398 | 0.97285 | 2.6580 | 2.3855 | [illegible] |
| 0.25 | 4.0 | 3.88814 | 0.86804 | 20.43304 | 20.01983 | 1.39598 | 0.79573 | 3.5960 | 2.0498 | 4.479[illegible] |
| 0.20 | 5.0 | 4.90646 | 0.85182 | 20.39100 | 19.82317 | 1.69400 | 0.78749 | 4.4469 | 2.0672 | 5.7600 |
| 0.13 | 7.5 | 7.23034 | 0.84527 | 20.90641 | 20.13050 | 1.20454 | 0.90780 | 3.4246 | 2.4015 | 8.553 |
| 0.10 | 10.0 | 9.50080 | 1.00165 | 19.98033 | 20.00600 | 1.52919 | 1.03912 | 3.4137 | 2.3197 | 9.4852 |
| 0.05 | 20.0 | 19.41312 | 1.10184 | 19.73651 | 19.91383 | 1.92041 | 1.58387 | 3.8973 | 3.2143 | 17.6188 |
| 0.02 | 50.0 | 48.98143 | 1.03013 | 21.30876 | 20.25166 | 1.81499 | 0.94129 | 3.9397 | 2.0432 | 47.5488 |
| 0.01 | 100.0 | 94.28543 | 0.89177 | 18.09177 | 20.07933 | 1.79411 | 0.78378 | 4.4986 | 1.9653 | 105.7279 |

TABLE C-4

**MONTE CARLO RESULTS, SAMPLE SIZE 10, POPULATION MEAN 25**

| Rela. Step $\bar{d}/\gamma$ | GAMMA Pop. $\gamma$ | GAMMA Sample $g_s$ | BC $g_L$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | M | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09732 | — | 25.00924 | 24.96000 | 0.31716 | 0.11275 | 7.0917 | — | — |
| 6.67 | 0.15 | 0.14196 | — | 25.00378 | 24.93333 | 0.31167 | 0.11563 | 4.6460 | — | — |
| 4.00 | 0.25 | 0.25532 | — | 25.02197 | 24.96500 | 0.38888 | 0.16530 | 3.4781 | — | — |
| 3.00 | 0.333 | 0.32016 | — | 25.00020 | 24.90833 | 0.42416 | 0.11454 | 2.8481 | — | — |
| 2.00 | 0.5 | 0.45769 | 0.26584 | 24.96479 | 24.72500 | 0.56135 | 0.25432 | 4.7218 | 2.1392 | 1.7217 |
| 1.43 | 0.7 | 0.69531 | 0.42139 | 24.97658 | 24.54500 | 0.75545 | 0.61373 | 4.0087 | 3.2567 | 1.6500 |
| 1.18 | 0.85 | 0.86484 | 0.54493 | 24.93685 | 24.46416 | 0.96991 | 0.91441 | 3.9799 | 3.7522 | 1.5871 |
| 1.00 | 1.0 | 0.92684 | 0.66514 | 24.97344 | 24.29666 | 0.82891 | 0.99762 | 2.7866 | 3.3538 | 1.3934 |
| 0.67 | 1.5 | 1.44522 | 0.90575 | 25.03658 | 23.81216 | 1.19229 | 1.49793 | 2.9434 | 3.6980 | 1.5956 |
| 0.50 | 2.0 | 1.96106 | 0.79466 | 24.94610 | 23.54916 | 1.36963 | 1.05602 | 3.8540 | 2.9715 | 2.4678 |
| 0.40 | 2.5 | 2.35669 | 1.08508 | 25.03229 | 23.3348[illegible] | 1.55674 | 1.72050 | 3.2080 | 3.5455 | 2.1995 |
| 0.33 | 3.0 | 2.88753 | 1.26185 | 25.09213 | 22.605[illegible] | 1.65964 | 1.64098 | 2.9410 | 2.9079 | 2.2883 |
| 0.25 | 4.0 | 3.76046 | 1.31152 | 25.40623 | 22.30[illegible] | 1.55414 | 1.70031 | 2.6497 | 2.8989 | 2.8673 |
| 0.20 | 5.0 | 4.62435 | 1.27991 | 25.50666 | 22.171[illegible] | 2.00507 | 1.88906 | 3.5030 | 3.3003 | 3.6130 |
| 0.13 | 7.5 | 7.12400 | 1.42097 | 24.55048 | 21.1898[illegible] | 1.74465 | 2.04690 | 2.7454 | 3.2210 | 5.0135 |
| 0.10 | 10.0 | 9.56562 | 0.97078 | 25.64606 | 20.87475 | 2.68375 | 1.08556 | 6.1817 | 2.5004 | 9.8535 |
| 0.05 | 20.0 | 20.23556 | 0.75837 | 24.33912 | 20.32016 | 1.64375 | 0.85664 | 4.8466 | 2.5258 | 26.6831 |
| 0.02 | 50.0 | 48.48879 | 1.00019 | 26.49320 | 20.28283 | 1.88021 | 1.18542 | 4.2035 | 2.6502 | 48.4794 |
| 0.01 | 100.0 | 91.01865 | 1.06043 | 37.28975 | 20.67916 | 1.93027 | 1.37223 | 4.0703 | 2.8936 | 85.8321 |

TABLE C-5

MONTE CARLO RESULTS, SAMPLE SIZE 20, POPULATION MEAN 19.5

| Rela. Step d/γ | GAMMA Pop. γ | | | MEAN Sample $m_s$ | | STD. DEV. Mean $s_m$ | | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. γ | Sample $g_s$ | BCTN $g_B$ | Sample $m_s$ | BCTN $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | |
| 10.00 | 0.1 | 0.09980 | 0.09648 | 19.50120 | 19.49300 | 0.03294 | 0.02122 | 1.0277 | 0.6620 | 1.0346 |
| 6.67 | 0.15 | 0.15045 | 0.14241 | 19.50551 | 19.51689 | 0.08695 | 0.05558 | 1.9272 | 1.2319 | 1.0565 |
| 4.00 | 0.25 | 0.24780 | 0.23678 | 19.49736 | 19.48577 | 0.14756 | 0.10332 | 2.0579 | 1.4479 | 1.0465 |
| 3.00 | 0.333 | 0.34210 | 0.31052 | 19.50496 | 19.53644 | 0.22010 | 0.12023 | 2.4259 | 1.3252 | 1.1017 |
| 2.00 | 0.5 | 0.47978 | 0.39933 | 19.52259 | 19.55800 | 0.26383 | 0.19980 | 2.0892 | 1.5822 | 1.2015 |
| 1.43 | 0.7 | 0.69940 | 0.54156 | 19.48672 | 19.54311 | 0.41329 | 0.30354 | 2.4133 | 1.7724 | 1.2915 |
| 1.18 | 0.85 | 0.83854 | 0.60159 | 19.49230 | 19.49966 | 0.40136 | 0.30271 | 2.1098 | 1.5912 | 1.3939 |
| 1.00 | 1.0 | 0.95525 | 0.71625 | 19.44537 | 19.54222 | 0.49057 | 0.39982 | 2.1599 | 1.7603 | 1.3300 |
| 0.67 | 1.5 | 1.48774 | 0.86370 | 19.51593 | 19.59627 | 0.65270 | 0.50598 | 2.3397 | 1.8526 | 1.7225 |
| 0.50 | 2.0 | 1.95833 | 1.14393 | 19.50337 | 19.68691 | 0.90781 | 0.73161 | 2.5096 | 2.0225 | 1.7119 |
| 0.40 | 2.5 | 2.44134 | 1.18547 | 19.59706 | 19.73894 | 0.97918 | 0.62093 | 2.6120 | 1.6563 | 2.0594 |
| 0.33 | 3.0 | 2.98332 | 1.50144 | 19.46146 | 19.55458 | 1.21924 | 1.05961 | 2.5679 | 2.2317 | 1.9870 |
| 0.25 | 4.0 | 3.87915 | 1.55172 | 19.48446 | 19.71027 | 1.12369 | 1.07033 | 2.2900 | 2.1812 | 2.4999 |
| 0.20 | 5.0 | 5.04725 | 1.81860 | 19.71790 | 19.73109 | 1.38009 | 1.71743 | 2.3998 | 2.9863 | 2.7753 |
| 0.13 | 7.5 | 7.13063 | 1.82087 | 19.37840 | 19.62066 | 1.82317 | 1.33204 | 3.1663 | 2.3133 | 3.9161 |
| 0.10 | 10.0 | 9.71018 | 2.19494 | 19.91108 | 20.09454 | 2.09121 | 1.80466 | 3.0128 | 2.6000 | 4.4239 |
| 0.05 | 20.0 | 20.21420 | 2.33443 | 18.57473 | 19.67988 | 2.26066 | 2.01202 | 3.0623 | 2.7255 | 8.6591 |
| 0.02 | 50.0 | 48.78458 | 2.65181 | 16.10018 | 19.48389 | 2.85919 | 2.07512 | 3.4096 | 2.4746 | 18.3967 |
| 0.01 | 100.0 | 99.86594 | 2.97913 | 22.79567 | 19.89671 | 2.70499 | 2.39000 | 2.8713 | 2.5369 | 33.5216 |

TABLE C-6

**MONTE CARLO RESULTS, SAMPLE SIZE 20, POPULATION MEAN 20**

| Rela. Step d/γ | Pop. γ | GAMMA Sample $g_s$ | BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10183 | — | 19.99501 | 20.00100 | 0.15933 | 0.03090 | 2.2205 | — | — |
| 6.67 | 0.15 | 0.15011 | — | 19.98980 | 19.96700 | 0.14232 | 0.03045 | 1.9662 | — | — |
| 4.00 | 0.25 | 0.24657 | — | 19.99885 | 20.00866 | 0.17409 | 0.08816 | 2.1015 | — | — |
| 3.00 | 0.333 | 0.32485 | — | 19.96928 | 19.97555 | 0.19433 | 0.11299 | 2.2422 | — | — |
| 2.00 | 0.5 | 0.49339 | 0.41324 | 20.02590 | 20.02288 | 0.25126 | 0.21512 | 1.9227 | 1.6462 | 1.1954 |
| 1.43 | 0.7 | 0.67992 | 0.53948 | 19.99950 | 19.97577 | 0.33265 | 0.23278 | 1.9499 | 1.3645 | 1.2603 |
| 1.18 | 0.85 | 0.84055 | 0.69331 | 20.03448 | 20.00700 | 0.42750 | 0.41059 | 1.9499 | 1.8727 | 1.2124 |
| 1.00 | 1.0 | 0.94833 | 0.71035 | 19.92713 | 19.91044 | 0.40244 | [illegible].1014 | 1.6385 | 0.9877 | 0.99916 |
| 0.67 | 1.5 | 1.48041 | 0.90870 | 20.01985 | 19.98373 | 0.59944 | 0.53693 | 2.0860 | 1.8685 | 1.6292 |
| 0.50 | 2.0 | 1.99297 | 1.04865 | 20.01285 | 19.94252 | 0.91202 | 0.56035 | 2.7503 | 1.6898 | 1.9005 |
| 0.40 | 2.5 | 2.44934 | 1.28425 | 19.97267 | 19.95089 | 0.96141 | 0.92147 | 2.3673 | 2.2690 | 1.9072 |
| 0.33 | 3.0 | 2.92165 | 1.44052 | 19.86653 | 19.91763 | 0.96700 | 0.93174 | 2.1228 | 2.0454 | 2.0282 |
| 0.25 | 4.0 | 3.98349 | 1.63132 | 19.92317 | 20.02348 | 1.35756 | 1.16517 | 2.6316 | 2.2587 | 2.4419 |
| 0.20 | 5.0 | 4.93130 | 1.43451 | 19.97791 | 19.91954 | 1.42400 | 0.90234 | 3.1391 | 1.9891 | 3.4376 |
| 0.13 | 7.5 | 7.55282 | 2.08836 | 20.13699 | 20.19076 | 1.99238 | 1.63438 | 3.0169 | 2.4748 | 3.6166 |
| 0.10 | 10.0 | 9.94247 | 1.87517 | 19.50612 | 19.99982 | 1.68507 | 1.42813 | 2.8417 | 2.4084 | 5.3022 |
| 0.05 | 20.0 | 19.61793 | 2.58116 | 19.96075 | 20.04885 | 2.23382 | 2.86215 | 2.7367 | 3.5065 | 7.6004 |
| 0.02 | 50.0 | 47.36463 | [illegible].09673 | 19.81488 | 20.16439 | 2.33970 | 3.49268 | 2.3892 | 3.5666 | 15.2950 |
| 0.01 | 100.0 | 99.57280 | 2.51279 | 20.88535 | 20.10244 | 2.72513 | 2.12304 | 3.4295 | 2.6718 | 39.6263 |

TABLE C-7

**MONTE CARLO RESULTS, SAMPLE SIZE 20, POPULATION MEAN 20.5**

| Relative Step $d/\gamma$ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. $\gamma$ | Sample $g_s$ | BCTN $g_B$ | Sample $m_s$ | BCTN $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | $g_s/g_B$ |
| 10.00 | 0.1 | 0.09673 | 0.10136 | 20.49711 | 20.49600 | 0.04272 | 0.02939 | 1.3328 | 0.9169 | 0.9543 |
| 6.67 | 0.15 | 0.14623 | 0.14267 | 20.49710 | 20.48778 | 0.07877 | 0.05072 | 1.7459 | 1.1242 | 1.0250 |
| 4.00 | 0.25 | 0.24244 | 0.22674 | 20.49886 | 20.50922 | 0.15048 | 0.08504 | 2.0986 | 1.1860 | 1.0692 |
| 3.00 | 0.333 | 0.32869 | 0.28690 | 20.52243 | 20.50588 | 0.20103 | 0.09426 | 2.2158 | 1.0389 | 1.1457 |
| 2.00 | 0.5 | 0.47829 | 0.43092 | 20.47389 | 20.44244 | 0.29143 | 0.20767 | 2.1386 | 1.5239 | 1.1099 |
| 1.43 | 0.7 | 0.68242 | 0.55632 | 20.45537 | 20.36755 | 0.35767 | 0.36356 | 2.0331 | 2.0666 | 1.2267 |
| 1.18 | 0.85 | 0.81027 | 0.69705 | 20.45229 | 20.38866 | 0.46992 | 0.38938 | 2.1319 | 1.7665 | 1.1624 |
| 1.00 | 1.0 | 0.95394 | 0.66981 | 20.50222 | 20.47344 | 0.43746 | 0.22324 | 2.0653 | 1.5260 | 1.4242 |
| 0.67 | 1.5 | 1.47202 | 0.90366 | 20.54754 | 20.54002 | 0.61361 | 0.50348 | 2.1473 | 1.7619 | 1.6289 |
| 0.50 | 2.0 | 1.94863 | 1.28408 | 20.46891 | 20.40777 | 0.93877 | 0.72163 | 2.3119 | 1.7771 | 1.5175 |
| 0.40 | 2.5 | 2.39505 | 1.18413 | 20.52999 | 20.51147 | 0.99947 | 0.86980 | 2.6691 | 2.3228 | 2.0226 |
| 0.33 | 3.0 | 2.95405 | 1.51427 | 20.70671 | 20.53578 | 1.18711 | 1.04758 | 2.4791 | 2.1877 | 1.9508 |
| 0.25 | 4.0 | 3.91696 | 1.54782 | 20.57154 | 20.16480 | 1.49149 | 1.09683 | 3.0472 | 2.2409 | 2.5306 |
| 0.20 | 5.0 | 4.95455 | 1.61571 | 20.55937 | 20.35456 | 1.38637 | 1.40576 | 2.7134 | 2.7514 | 3.0665 |
| 0.13 | 7.5 | 7.38692 | 1.90461 | 19.93791 | 19.96735 | 1.86903 | 1.57327 | 3.1032 | 2.6121 | 3.8784 |
| 0.10 | 10.0 | 9.61840 | 1.97545 | 20.59826 | 20.19554 | 2.25101 | 1.66688 | 3.6034 | 2.6683 | 4.8690 |
| 0.05 | 20.0 | 19.04328 | 2.83280 | 20.76487 | 20.32276 | 2.29794 | 3.00861 | 2.5652 | 3.3585 | 6.7224 |
| 0.02 | 50.0 | 49.32339 | 2.61272 | 21.89346 | 20.43539 | 2.55257 | 2.14128 | 3.0895 | 2.5917 | 18.8781 |
| 0.01 | 100.0 | 97.05620 | 2.90003 | 20.94858 | 20.08282 | 2.65374 | 2.74642 | 2.8937 | 2.9948 | 33.4673 |

TABLE C-8

MONTE CARLO RESULTS, SAMPLE SIZE 20, POPULATION MEAN 25

| Relative Step $d/\gamma$ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. $\gamma$ | Sample $g_s$ | BCTN $g_B$ | Sample $m_s$ | BCTN $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | $g_s/g_B$ |
| 10.00 | 0.1 | 0.09887 | — | 25.00486 | 24.99017 | 0.15465 | 0.03690 | 2.1605 | — | — |
| 6.67 | 0.15 | 0.15139 | — | 24.99888 | 24.97482 | 0.18643 | 0.05204 | 2.7094 | — | — |
| 4.00 | 0.25 | 0.24404 | — | 25.00155 | 24.91857 | 0.20973 | 0.08362 | 2.7605 | — | — |
| 3.00 | 0.333 | 0.32435 | — | 24.99889 | 24.95946 | 0.21455 | 0.11252 | 2.3988 | — | — |
| 2.00 | 0.5 | 0.48513 | 0.42583 | 25.00035 | 24.91446 | 0.30074 | 0.25476 | 2.2333 | 1.8918 | 1.1392 |
| 1.43 | 0.7 | 0.69427 | 0.61504 | 25.04535 | 24.93920 | 0.41700 | 0.43239 | 2.1441 | 2.2232 | 1.1288 |
| 1.18 | 0.85 | 0.78521 | 0.69320 | 24.99897 | 24.79706 | 0.47967 | 0.49449 | 2.1882 | 2.2558 | 1.1327 |
| 1.00 | 1.0 | 1.01217 | 1.00938 | 24.99277 | 24.81932 | 0.53945 | 0.77482 | 1.6900 | 2.4275 | 1.0028 |
| 0.67 | 1.5 | 1.49770 | 1.48082 | 24.91449 | 24.53123 | 0.75674 | 1.14891 | 1.6160 | 2.4535 | 1.0114 |
| 0.50 | 2.0 | 1.95705 | 1.91662 | 25.10433 | 24.40117 | 1.01404 | 1.46841 | 1.6731 | 2.4228 | 1.0211 |
| 0.40 | 2.5 | 2.44462 | 2.17279 | 24.82331 | 23.82718 | 1.24064 | 1.71554 | 1.8056 | 2.4968 | 1.1251 |
| 0.33 | 3.0 | 2.94416 | 2.02323 | 24.73553 | 23.64899 | 1.30524 | 1.46240 | 2.0401 | 2.2857 | 1.4552 |
| 0.25 | 4.0 | 3.95674 | 2.92533 | 25.30142 | 23.47872 | 1.47685 | 2.78457 | 1.5965 | 3.0101 | 1.3526 |
| 0.20 | 5.0 | 5.01318 | 2.47808 | 25.09871 | 23.19057 | 1.84156 | 1.77313 | 2.3500 | 2.2627 | 2.0230 |
| 0.13 | 7.5 | 7.28622 | 2.90864 | 25.04321 | 22.39888 | 1.93256 | 2.44949 | 2.1011 | 2.6631 | 2.5050 |
| 0.10 | 10.0 | 10.02730 | 3.20819 | 24.63659 | 21.83936 | 1.92033 | 3.16211 | 1.8928 | 3.1169 | 3.1255 |
| 0.05 | 20.0 | 19.62157 | 2.84951 | 26.59866 | 21.29793 | 2.06288 | 3.20410 | 2.2893 | 3.5558 | 6.8859 |
| 0.02 | 50.0 | 49.37360 | 2.53170 | 27.22243 | 20.86003 | 2.19625 | 2.33026 | 2.7433 | 2.3107 | 19.5022 |
| 0.01 | 100.0 | 98.99462 | 2.65004 | 31.78864 | 19.99277 | 2.56356 | 3.03889 | 3.0591 | 3.6263 | 37.3559 |

TABLE C-9

**MONTE CARLO RESULTS, SAMPLE SIZE 50, POPULATION MEAN 19.5**

| Relative Step d/γ | Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09913 | 0.09738 | 19.50132 | 19.49760 | 0.02329 | 0.01406 | 1.1834 | 0.7144 | 1.0180 |
| 6.67 | 0.15 | 0.14913 | 0.14386 | 19.49878 | 19.50801 | 0.05649 | 0.03719 | 1.9634 | 1.2926 | 1.0366 |
| 4.00 | 0.25 | 0.24548 | 0.23335 | 19.49355 | 19.48341 | 0.09181 | 0.04929 | 1.9672 | 1.0561 | 1.0520 |
| 3.00 | 0.333 | 0.32814 | 0.31958 | 19.49010 | 19.51850 | 0.12633 | 0.07502 | 1.9766 | 1.1738 | 1.0269 |
| 2.00 | 0.5 | 0.49027 | 0.45277 | 19.49584 | 19.49983 | 0.17879 | 0.14118 | 1.9744 | 1.5591 | 1.0828 |
| 1.43 | 0.7 | 0.68790 | 0.66002 | 19.50117 | 19.47415 | 0.23687 | 0.22728 | 1.7944 | 1.7218 | 1.0422 |
| 1.18 | 0.85 | 0.84268 | 0.76576 | 19.52336 | 19.50993 | 0.24320 | 0.26015 | 1.5880 | 1.6986 | 1.1005 |
| 1.00 | 1.0 | 0.98113 | 0.83474 | 19.56999 | 19.55421 | 0.31151 | 0.27516 | 1.8659 | 1.6482 | 1.1754 |
| 0.67 | 1.5 | 1.49494 | 1.26691 | 19.43569 | 19.41644 | 0.44383 | 0.51989 | 1.7516 | 2.0518 | 1.1800 |
| 0.50 | 2.0 | 1.97093 | 1.56708 | 19.47467 | 19.50393 | 0.55306 | 0.63663 | 1.7646 | 2.0313 | 1.2577 |
| 0.40 | 2.5 | 2.49932 | 1.94965 | 19.47893 | 19.54273 | 0.65458 | 0.98919 | 1.6787 | 2.5368 | 1.2819 |
| 0.33 | 3.0 | 2.98845 | 2.16650 | 19.55879 | 19.50960 | 0.86761 | 1.16993 | 2.0023 | 2.7000 | 1.3794 |
| 0.25 | 4.0 | 4.03309 | 2.72268 | 19.43149 | 19.70047 | 1.01882 | 1.51334 | 1.8710 | 2.7791 | 1.4813 |
| 0.20 | 5.0 | 5.08339 | 3.49379 | 19.54627 | 19.75091 | 1.30972 | 2.20470 | 1.8744 | 3.1552 | 1.4550 |
| 0.13 | 7.5 | 7.46359 | 4.15730 | 19.28108 | 19.68664 | 1.53672 | 2.89277 | 1.8482 | 3.4791 | 1.7953 |
| 0.10 | 10.0 | 9.80214 | 4.10262 | 19.19168 | 19.33641 | 1.80950 | 2.73067 | 2.2053 | 3.3280 | 2.3892 |
| 0.05 | 20.0 | 19.64408 | 4.60801 | 20.24534 | 20.16889 | 3.00836 | 3.12486 | 3.2643 | 3.3907 | 4.2630 |
| 0.02 | 50.0 | 50.71653 | 5.70694 | 19.11390 | 19.81490 | 3.34219 | 4.41472 | 2.9282 | 3.8679 | 8.8868 |
| 0.01 | 100.0 | 99.86888 | 6.85854 | 16.23517 | 19.22018 | 4.41727 | 5.89254 | 3.2203 | 4.2958 | 14.5613 |

TABLE C-10

**MONTE CARLO RESULTS, SAMPLE SIZE 50, POPULATION MEAN 20**

| Relative Step d/γ | GAMMA Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTM $g_B$ | MEAN Sample $m_s$ | MEAN BCTM $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09940 | — | 19.99886 | 20.00159 | 0.10075 | 0.01527 | 2.0916 | — | — |
| 6.67 | 0.15 | 0.15056 | — | 19.99995 | 20.00480 | 0.10060 | 0.01735 | 2.0817 | — | — |
| 4.00 | 0.25 | 0.24992 | — | 20.00012 | 20.00340 | 0.10683 | 0.04595 | 2.0024 | — | — |
| 3.00 | 0.333 | 0.33463 | — | 19.98521 | 19.97086 | 0.11184 | 0.06907 | 1.7036 | — | — |
| 2.00 | 0.5 | 0.49548 | 0.45538 | 20.01685 | 20.02041 | 0.16079 | 0.12039 | 1.7655 | 1.3219 | 1.0881 |
| 1.43 | 0.7 | 0.69143 | 0.65021 | 19.99373 | 20.00886 | 0.23036 | 0.21536 | 1.7714 | 1.6561 | 1.0634 |
| 1.18 | 0.85 | 0.84952 | 0.81588 | 20.00261 | 20.00970 | 0.27463 | 0.25965 | 1.6830 | 1.5912 | 1.0412 |
| 1.00 | 1.0 | 0.99313 | 0.88791 | 19.97644 | 19.95633 | 0.31199 | 0.33632 | 1.7569 | 1.8939 | 1.1185 |
| 0.67 | 1.5 | 1.48251 | 1.28916 | 19.97018 | 19.99103 | 0.46634 | 0.50088 | 1.8687 | 1.9427 | 1.1500 |
| 0.50 | 2.0 | 1.98361 | 1.53513 | 20.00536 | 19.99374 | 0.63847 | 0.75414 | 2.0795 | 2.4563 | 1.2921 |
| 0.40 | 2.5 | 2.46923 | 1.90750 | 19.95094 | 20.00003 | 0.71253 | 0.94143 | 1.8677 | 2.4677 | 1.2945 |
| 0.33 | 3.0 | 2.99865 | 2.10387 | 20.04514 | 20.12826 | 0.78130 | 1.02550 | 1.8658 | 2.4372 | 1.4253 |
| 0.25 | 4.0 | 4.03867 | 2.46214 | 19.99438 | 19.93305 | 1.10604 | 1.38625 | 2.2461 | 2.8151 | 1.6403 |
| 0.20 | 5.0 | 4.94970 | 3.18083 | 19.95613 | 19.96903 | 1.33784 | 1.71140 | 2.1030 | 2.6902 | 1.5561 |
| 0.13 | 7.5 | 7.53555 | 3.39446 | 20.22657 | 19.99674 | 1.73467 | 1.71914 | 2.5551 | 2.5323 | 2.2200 |
| 0.10 | 10.0 | 10.26661 | 4.48515 | 19.78273 | 19.68458 | 2.11882 | 3.03699 | 2.3620 | 3.3856 | 2.2890 |
| 0.05 | 20.0 | 19.77755 | 4.61543 | 19.92211 | 20.09712 | 2.56438 | 3.02759 | 2.7780 | 3.2799 | 4.2851 |
| 0.02 | 50.0 | 50.55966 | 5.98699 | 19.22602 | 19.91470 | 3.33666 | 5.24302 | 2.7866 | 4.3787 | 8.4149 |
| 0.01 | 100.0 | 99.04810 | 7.58783 | 16.89225 | 19.30135 | 4.69949 | 7.16879 | 3.0967 | 4.7239 | 13.0536 |

TABLE C-11

**MONTE CARLO RESULTS, SAMPLE SIZE 50, POPULATION MEAN 20.5**

| Relative Step $d/\gamma$ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. $\gamma$ | Sample $g_s$ | BCTM $g_B$ | Sample $m_s$ | BCTM $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | $g_s/g_B$ |
| 10.00 | 0.1 | 0.10054 | 0.09883 | 20.50352 | 20.49720 | 0.02119 | 0.01585 | 1.0767 | 0.8054 | 1.0173 |
| 6.67 | 0.15 | 0.14937 | 0.13970 | 20.50175 | 20.49806 | 0.04926 | 0.03200 | 1.7631 | 1.1453 | 1.0692 |
| 4.00 | 0.25 | 0.24630 | 0.23191 | 20.49988 | 20.48586 | 0.09604 | 0.05426 | 2.0706 | 1.1698 | 1.0620 |
| 3.00 | 0.333 | 0.33574 | 0.30984 | 20.50184 | 20.49446 | 0.12934 | 0.06024 | 2.0872 | 0.9721 | 1.0836 |
| 2.00 | 0.5 | 0.48645 | 0.47437 | 20.47327 | 20.48440 | 0.17776 | 0.14234 | 1.8737 | 1.5003 | 1.0255 |
| 1.43 | 0.7 | 0.69271 | 0.63830 | 20.49479 | 20.45628 | 0.25476 | 0.22078 | 1.9956 | 1.7294 | 1.0852 |
| 1.18 | 0.85 | 0.86317 | 0.73595 | 20.48988 | 20.51080 | 0.28121 | 0.23416 | 1.9105 | 1.5909 | 1.1729 |
| 1.00 | 1.0 | 1.00229 | 0.89080 | 20.50753 | 20.45676 | 0.27393 | 0.34575 | 1.5375 | 1.9407 | 1.1252 |
| 0.67 | 1.5 | 1.51274 | 1.21581 | 20.56972 | 20.52562 | 0.43339 | 0.46491 | 1.7823 | 1.9119 | 1.2442 |
| 0.50 | 2.0 | 1.99699 | 1.62704 | 20.45599 | 20.36939 | 0.50690 | 0.67774 | 1.5577 | 2.0827 | 1.2274 |
| 0.40 | 2.5 | 2.42135 | 1.80498 | 20.44850 | 20.40793 | 0.80322 | 0.81066 | 2.2250 | 2.2456 | 1.3415 |
| 0.33 | 3.0 | 3.00047 | 2.17734 | 20.41161 | 20.39293 | 0.72641 | 1.24350 | 1.6681 | 2.8556 | 1.3780 |
| 0.25 | 4.0 | 3.86687 | 2.41833 | 20.42355 | 20.39294 | 1.01119 | 1.13495 | 2.0907 | 2.3466 | 1.5990 |
| 0.20 | 5.0 | 5.01661 | 3.29418 | 20.44514 | 20.47041 | 1.23653 | 1.73648 | 1.8768 | 2.6357 | 1.5229 |
| 0.13 | 7.5 | 7.53869 | 3.39811 | 20.56903 | 20.32529 | 1.66443 | 1.91211 | 2.4491 | 2.8135 | 2.2185 |
| 0.10 | 10.0 | 9.91544 | 4.09487 | 20.63836 | 20.30181 | 2.00610 | 2.79651 | 2.4495 | 3.4146 | 2.4214 |
| 0.05 | 20.0 | 19.87332 | 4.17690 | 21.28222 | 20.56620 | 2.55643 | 2.81054 | 3.0602 | 3.3644 | 4.7579 |
| 0.02 | 50.0 | 49.83894 | 6.29097 | 20.82349 | 20.48378 | 3.75580 | 5.46804 | 2.9851 | 4.3459 | 7.9223 |
| 0.01 | 100.0 | 100.34205 | 7.28908 | 21.67330 | 20.21938 | 3.78476 | 7.07686 | 2.5962 | 4.8544 | 13.7661 |

TABLE C-12

**MONTE CARLO RESULTS, SAMPLE SIZE 50, POPULATION MEAN 25**

| Relative Step $d/\gamma$ | GAMMA Pop. $\gamma$ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10007 | — | 25.00450 | 24.98600 | 0.10583 | 0.01712 | 2.2148 | — | — |
| 6.67 | 0.15 | 0.14911 | — | 25.00565 | 25.00774 | 0.10807 | 0.02224 | 2.2399 | — | — |
| 4.00 | 0.25 | 0.24573 | — | 24.99964 | 24.99885 | 0.11735 | 0.05220 | 2.1761 | — | — |
| 3.00 | 0.333 | 0.33315 | — | 25.01448 | 25.00181 | 0.13290 | 0.08504 | 1.9241 | — | — |
| 2.00 | 0.5 | 0.50965 | 0.50542 | 25.00938 | 24.95934 | 0.18510 | 0.15623 | 1.8312 | 1.5455 | 1.0084 |
| 1.43 | 0.7 | 0.70959 | 0.66634 | 24.97845 | 24.95109 | 0.26228 | 0.23690 | 1.9681 | 1.7776 | 1.0649 |
| 1.18 | 0.85 | 0.85547 | 0.89613 | 24.99086 | 24.92258 | 0.30783 | 0.30503 | 1.7176 | 1.7019 | 0.9546 |
| 1.00 | 1.0 | 0.97943 | 0.99511 | 24.99889 | 24.93250 | 0.34156 | 0.39615 | 1.7162 | 1.9905 | 0.9842 |
| 0.67 | 1.5 | 1.49229 | 1.39735 | 25.03622 | 24.87077 | 0.46796 | 0.49283 | 1.6745 | 1.7634 | 1.0679 |
| 0.50 | 2.0 | 1.98653 | 2.10142 | 24.99615 | 24.77719 | 0.58351 | 0.87206 | 1.3884 | 2.0749 | 0.9453 |
| 0.40 | 2.5 | 2.46089 | 2.49958 | 24.94832 | 24.65241 | 0.71119 | 1.14346 | 1.4226 | 2.2873 | 0.9845 |
| 0.33 | 3.0 | 3.02135 | 2.98463 | 24.93018 | 24.46717 | 0.85062 | 1.55011 | 1.4250 | 2.5968 | 1.0123 |
| 0.25 | 4.0 | 3.92[illegible]04 | 3.74057 | 25.02917 | 24.35528 | 1.19780 | 2.05156 | 1.6011 | 2.7423 | 1.0488 |
| 0.20 | 5.0 | 5.03784 | 4.26573 | 24.90073 | 24.07883 | 1.29221 | 2.84338 | 1.5146 | 3.3328 | 1.1810 |
| 0.13 | 7.5 | 7.41918 | 5.83011 | 25.13256 | 23.68848 | 1.65407 | 4.04707 | 1.4186 | 3.4708 | 1.2726 |
| 0.10 | 10.0 | 9.88829 | 5.71626 | 24.90527 | 23.35131 | 1.95762 | 3.74108 | 1.7123 | 3.2723 | 1.7299 |
| 0.05 | 20.0 | 19.79714 | 6.99948 | 24.73659 | 22.65854 | 2.72375 | 6.21155 | 1.9457 | 4.4372 | 2.8284 |
| 0.02 | 50.0 | 49.47536 | 5.89807 | 25.43928 | 21.30729 | 3.34475 | 4.27329 | 2.8355 | 3.6226 | 8.3884 |
| 0.01 | 100.0 | 100.44436 | 6.96501 | 24.73240 | 21.31166 | 4.06526 | 6.19837 | 2.9183 | 4.4497 | 14.4213 |

TABLE C-13

**MONTE CARLO RESULTS, SAMPLE SIZE 100, POPULATION MEAN 19.5**

| Relative Step d/γ | GAMMA Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_p$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09839 | 0.09906 | 19.49992 | 19.49840 | 0.01647 | 0.01105 | 1.1756 | 0.7988 | 0.9932 |
| 6.67 | 0.15 | 0.14683 | 0.14388 | 19.50624 | 19.50360 | 0.03560 | 0.02501 | 1.7496 | 1.2291 | 1.0205 |
| 4.00 | 0.25 | 0.24942 | 0.24667 | 19.49634 | 19.50061 | 0.07786 | 0.03915 | 2.2319 | 1.1223 | 1.0111 |
| 3.00 | 0.33 | 0.33915 | 0.32421 | 19.50668 | 19.50620 | 0.08425 | 0.05248 | 1.8375 | 1.1446 | 1.0461 |
| 2.00 | 0.5 | 0.49693 | 0.47772 | 19.49758 | 19.51219 | 0.12767 | 0.11173 | 1.8898 | 1.6539 | 1.0402 |
| 1.43 | 0.7 | 0.69328 | 0.64988 | 19.51228 | 19.51791 | 0.15492 | 0.13606 | 1.6856 | 1.4804 | 1.0746 |
| 1.18 | 0.85 | 0.84662 | 0.81480 | 19.51551 | 19.54729 | 0.19807 | 0.16396 | 1.7169 | 1.4229 | 1.0391 |
| 1.00 | 1.0 | 1.01709 | 0.96564 | 19.47698 | 19.49595 | 0.28333 | 0.19875 | 1.7452 | 1.4554 | 1.0533 |
| 0.67 | 1.5 | 1.51670 | 1.40475 | 19.50742 | 19.49932 | 0.37931 | 0.45438 | 1.9093 | 2.2872 | 1.0797 |
| 0.50 | 3.0 | 1.99735 | 1.75043 | 19.42967 | 19.44133 | 0.40569 | 0.52167 | 1.6388 | 2.1974 | 1.1411 |
| 0.40 | 2.5 | 2.48052 | 2.22609 | 19.47503 | 19.49937 | 0.50887 | 0.67631 | 1.6164 | 2.1493 | 1.1143 |
| 0.33 | 3.0 | 2.98217 | 2.49908 | 19.46525 | 19.54193 | 0.68580 | 0.88006 | 1.9405 | 2.4901 | 1.1933 |
| 0.25 | 4.0 | 4.01391 | 3.15132 | 19.55985 | 19.54331 | 0.78579 | 1.33294 | 1.7632 | 2.9909 | 1.2734 |
| 0.20 | 5.0 | 4.91580 | 3.69667 | 19.53383 | 19.47321 | 0.88554 | 1.51497 | 1.6939 | 2.8979 | 1.3298 |
| 0.13 | 7.5 | 7.36544 | 5.65648 | 19.45714 | 19.45927 | 1.53489 | 2.94476 | 1.9187 | 3.6812 | 1.3021 |
| 0.10 | 10.0 | 9.83046 | 6.00364 | 19.71283 | 19.87300 | 1.44272 | 2.86734 | 1.6992 | 3.3771 | 1.6474 |
| 0.05 | 20.0 | 19.83580 | 3.77243 | 19.47593 | 19.66606 | 2.94221 | 6.37590 | 2.3716 | 5.1401 | 2.2612 |
| 0.02 | 50.0 | 49.94327 | 12.27885 | 17.69949 | 19.10792 | 4.49124 | 10.25065 | 2.5864 | 5.9031 | 4.0674 |
| 0.01 | 100.0 | 98.19744 | 13.54787 | 20.71042 | 20.17384 | 5.13835 | 9.63563 | 2.6819 | 5.0291 | 7.2492 |

TABLE C-14

MONTE CARLO RESULTS, SAMPLE SIZE 100, POPULATION MEAN 20

| Relative Step d/γ | GAMMA Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTM $g_B$ | MEAN Sample $m_s$ | MEAN BCTM $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.09944 | — | 19.99356 | 19.99380 | 0.07992 | 0.00944 | — | — | — |
| 6.67 | 0.15 | 0.14978 | — | 20.00501 | 20.00560 | 0.08295 | 0.01012 | — | — | — |
| 4.00 | 0.25 | 0.24997 | — | 19.99480 | 20.00258 | 0.08050 | 0.04322 | — | — | — |
| 3.00 | 0.333 | 0.33212 | 0.33346 | 20.00776 | 19.99941 | 0.08468 | 0.05053 | 1.7957 | 1.0714 | 0.9960 |
| 2.00 | 0.5 | 0.49899 | 0.45719 | 19.99688 | 20.01297 | 0.12341 | 0.10030 | 1.9087 | 1.5606 | 1.0914 |
| 1.43 | 0.7 | 0.68047 | 0.65412 | 20.00803 | 20.03224 | 0.14257 | 0.15884 | 1.5412 | 1.7171 | 1.0403 |
| 1.18 | 0.85 | 0.84344 | 0.79959 | 20.01221 | 20.01302 | 0.22922 | 0.17986 | 2.0271 | 1.5906 | 1.0548 |
| 1.00 | 1.0 | 0.98824 | 0.92932 | 19.97983 | 19.99477 | 0.23587 | 0.23169 | 1.7947 | 1.7629 | 1.0634 |
| 0.67 | 1.5 | 1.50317 | 1.49951 | 20.01584 | 20.04942 | 0.31641 | 0.44572 | 1.4921 | 2.1018 | 1.0024 |
| 0.50 | 2.0 | 1.96859 | 1.65992 | 19.99731 | 19.95929 | 0.38041 | 0.50799 | 1.6205 | 2.1640 | 1.1860 |
| 0.40 | 2.5 | 2.50067 | 2.18457 | 19.98814 | 20.02076 | 0.49808 | 0.75737 | 1.6122 | 2.4515 | 1.1447 |
| 0.33 | 3.0 | 2.98321 | 2.44325 | 19.99722 | 19.97106 | 0.62923 | 0.85998 | 1.8209 | 2.4887 | 1.2209 |
| 0.25 | 4.0 | 4.03196 | 3.32643 | 20.00650 | 20.01105 | 0.88343 | 1.38150 | 1.8912 | 2.9376 | 1.2125 |
| 0.20 | 5.0 | 4.96360 | 3.73585 | 20.02116 | 20.00100 | 1.03860 | 1.70137 | 1.9658 | 3.2203 | 1.3286 |
| 0.13 | 7.5 | 7.37904 | 4.97579 | 19.99541 | 20.09986 | 1.52896 | 2.53388 | 2.1728 | 3.6009 | 1.4830 |
| 0.10 | 10.0 | 10.00000 | 6.52987 | 20.25166 | 20.25057 | 1.62563 | 3.40375 | 1.7604 | 3.6859 | 1.5314 |
| 0.67 | 15.0 | 14.83568 | 7.35814 | 20.10077 | 19.98469 | 2.55368 | 4.54776 | 2.4541 | 4.3703 | 2.0162 |
| 0.05 | 20.0 | 19.61782 | 7.88538 | 20.06179 | 19.87021 | 2.85224 | 4.82493 | 2.5577 | 4.3267 | 2.4879 |
| 0.02 | 50.0 | 49.47539 | 10.48738 | 18.77536 | 20.54055 | 4.28838 | 7.71192 | 2.8914 | 5.1997 | 4.7176 |
| 0.01 | 100.0 | 98.89081 | 13.14539 | 20.21219 | 20.61668 | 4.99743 | 9.88142 | 2.6882 | 5.3153 | 7.5229 |

TABLE C-15

**MONTE CARLO RESULTS, SAMPLE SIZE 100, POPULATION MEAN 20.5**

| Relative Step d/γ | GAMMA Pop. γ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10045 | 0.10326 | 20.49728 | 20.49860 | 0.01923 | 0.01333 | 1.3168 | 0.9128 | 0.9728 |
| 6.67 | 0.15 | 0.14788 | 0.14655 | 20.50188 | 20.49541 | 0.04016 | 0.02637 | 1.9379 | 1.2724 | 1.0091 |
| 4.00 | 0.25 | 0.24815 | 0.24489 | 20.50467 | 20.50319 | 0.07082 | 0.04118 | 2.0448 | 1.1890 | 1.0133 |
| 3.00 | 0.333 | 0.32914 | 0.32874 | 20.50292 | 20.49737 | 0.08430 | 0.04666 | 1.8132 | 1.0036 | 1.0012 |
| 2.00 | 0.5 | 0.48579 | 0.46373 | 20.50013 | 20.48542 | 0.12484 | 0.09993 | 1.9036 | 1.5238 | 1.0476 |
| 1.43 | 0.7 | 0.69366 | 0.65276 | 20.52598 | 20.51525 | 0.17080 | 0.13711 | 1.8502 | 1.4852 | 1.0627 |
| 1.18 | 0.85 | 0.83881 | 0.82382 | 20.51025 | 20.52027 | 0.18219 | 0.18586 | 1.5638 | 1.595[illegible] | 1.0182 |
| 1.00 | 1.0 | 0.98561 | 0.92395 | 20.50543 | 20.48146 | 0.22663 | 0.22338 | 1.7345 | 1.7095 | 1.0667 |
| 0.67 | 1.5 | 1.50097 | 1.40983 | 20.49998 | 20.47953 | 0.28983 | 0.41971 | 1.4537 | 2.1051 | 1.0646 |
| 0.50 | 2.0 | 2.00608 | 1.81196 | 20.48845 | 20.44896 | 0.38408 | 0.54107 | 1.4989 | 2.1115 | 1.1071 |
| 0.40 | 2.5 | 2.50605 | 2.18945 | 20.50259 | 20.49033 | 0.50162 | 0.77152 | 1.6201 | 2.4917 | 1.1446 |
| 0.33 | 3.0 | 3.00399 | 2.50771 | 20.47935 | 20.49363 | 0.69637 | 0.906[illegible] | 1.9636 | 2.5574 | 1.1979 |
| 0.25 | 4.0 | 3.94933 | 3.18288 | 20.51773 | 20.49821 | 0.78636 | 1.522[illegible] | 1.7470 | 3.3818 | 1.2408 |
| 0.20 | 5.0 | 4.99769 | 3.84462 | 20.54947 | 20.51807 | 0.86946 | 1.77904 | 1.5991 | 3.2720 | 1.2999 |
| 0.13 | 7.5 | 7.49169 | 4.96666 | 20.49678 | 20.46406 | 1.35768 | 2.57757 | 1.9329 | 3.6697 | 1.5084 |
| 0.10 | 10.0 | 10.05604 | 6.79196 | 20.71472 | 20.51938 | 1.9[illegible]321 | 3.62513 | 2.0335 | 3.7741 | 1.4806 |
| 0.05 | 20.0 | 19.79629 | 8.93359 | 20.67457 | 20.17674 | 2.87844 | 5.28591 | 2.2783 | 4.1839 | 2.2159 |
| 0.02 | 50.0 | 49.86611 | 11.39924 | 20.05274 | 19.92908 | 4.28132 | 7.44345 | 2.6557 | 4.6172 | 4.3745 |
| 0.01 | 100.0 | 98.51354 | 13.28621 | 17.93464 | 19.71010 | 4.38637 | 8.44728 | 2.3345 | 4.4957 | 7.4147 |

TABLE C-16

**MONTE CARLO RESULTS, SAMPLE SIZE 100, POPULATION MEAN 25**

| Relative Step d/γ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. γ | Sample $g_s$ | BCTH $g_B$ | Sample $m_s$ | BCTH $m_B$ | Mean $s_m$ | Gamma $g_\gamma$ | | | |
| 10.00 | 0.1 | 0.10000 | — | 24.99776 | 24.99551 | 0.7352 | 0.00705 | 2.1242 | — | — |
| 6.67 | 0.15 | 0.14915 | — | 25.00447 | 25.00040 | 0.07356 | 0.01044 | 2.1103 | — | — |
| 4.00 | 0.25 | 0.25311 | — | 25.00247 | 24.99179 | 0.08351 | 0.03733 | 2.0727 | — | — |
| 3.00 | 0.333 | 0.33073 | — | 24.99961 | 25.00226 | 0.08900 | 0.06055 | 1.8699 | — | — |
| 2.00 | 0.5 | 0.50099 | 0.48511 | 24.99814 | 24.98216 | 0.12435 | 0.10031 | 1.8125 | 1.4709 | 1.0327 |
| 1.43 | 0.7 | 0.69200 | 0.67842 | 25.02343 | 25.00774 | 0.15704 | 0.15728 | 1.6368 | 1.6393 | 1.0200 |
| 1.18 | 0.85 | 0.84717 | 0.83664 | 25.01693 | 24.99489 | 0.17937 | 0.20756 | 1.5160 | 1.7543 | 1.0126 |
| 1.00 | 1.00 | 1.00348 | 0.98743 | 24.99218 | 24.92537 | 0.22287 | 0.26011 | 1.5960 | 1.8627 | 1.0162 |
| 0.67 | 1.5 | 1.50069 | 1.51951 | 24.98329 | 24.90228 | 0.29911 | 0.45762 | 1.3919 | 2.1296 | 0.9876 |
| 0.50 | 2.0 | 2.00439 | 2.07605 | 25.00496 | 24.87074 | 0.45532 | 0.65724 | 1.5508 | 2.2386 | 0.9655 |
| 0.40 | 2.5 | 2.49308 | 2.51207 | 25.08034 | 24.91583 | 0.52678 | 0.98433 | 1.4828 | 2.7707 | 0.9924 |
| 0.33 | 3.0 | 2.98936 | 3.12824 | 25.00835 | 24.71540 | 0.53198 | 1.26205 | 1.2025 | 2.8527 | 0.9556 |
| 0.25 | 4.0 | 4.00679 | 4.12363 | 25.04434 | 24.83223 | 0.73029 | 1.86222 | 1.2523 | 3.1933 | 0.9717 |
| 0.20 | 5.0 | 4.99520 | 4.51550 | 24.91445 | 24.39058 | 1.02415 | 1.78865 | 1.6038 | 2.8009 | 1.1062 |
| 0.13 | 7.5 | 7.38421 | 6.05715 | 24.93072 | 24.17176 | 1.53553 | 2.84616 | 1.7926 | 3.3226 | 1.2191 |
| 0.10 | 10.0 | 10.01478 | 7.52331 | 24.73559 | 23.79591 | 1.75378 | 3.49253 | 1.6484 | 3.2826 | 1.3312 |
| 0.05 | 20.0 | 19.98829 | 10.66847 | 24.76263 | 23.26406 | 3.15429 | 7.24457 | 2.0907 | 4.8017 | 1.8736 |
| 0.02 | 50.0 | 50.18358 | 11.92343 | 24.01339 | 21.21386 | 4.39147 | 10.05573 | 2.6043 | 5.9634 | 4.2088 |
| 0.01 | 100.0 | 99.10170 | 13.36395 | 24.97929 | 20.73304 | 4.63603 | 12.33831 | 2.4530 | 6.5284 | 7.4156 |

TABLE C-17

MONTE CARLO RESULTS, SAMPLE SIZE 500, POPULATION MEAN 19.5

| Relative Step $d/\gamma$ | Gamma: Pop. $\gamma$ | Gamma: Sample $g_s$ | Gamma: BCTH $g_B$ | Mean: Sample $m_s$ | Mean: BCTH $m_B$ | Std. Dev.: Mean $s_m$ | Std. Dev.: Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10053 | 0.10043 | 19.50018 | 19.50040 | 0.01175 | 0.00515 | 1.8497 | 0.8107 | 1.0010 |
| 6.67 | 0.15 | 0.15011 | 0.14620 | 19.50092 | 19.50040 | 0.01871 | 0.01211 | 2.0233 | 1.3096 | 1.0267 |
| 4.00 | 0.25 | 0.25116 | 0.24920 | 19.50237 | 19.50400 | 0.03211 | 0.01908 | 2.0372 | 1.2105 | 1.0079 |
| 3.00 | 0.333 | 0.33187 | 0.33197 | 19.49895 | 19.49779 | 0.04255 | 0.02148 | 2.0264 | 1.0230 | 0.9997 |
| 2.00 | 0.5 | 0.49785 | 0.49845 | 19.50481 | 19.50651 | 0.06118 | 0.04444 | 1.9408 | 1.4095 | 0.9988 |
| 1.43 | 0.7 | 0.70407 | 0.69305 | 19.49992 | 19.50402 | 0.08336 | 0.06900 | 1.9019 | 1.5743 | 1.0159 |
| 1.18 | 0.85 | 0.84939 | 0.82728 | 19.50541 | 19.51005 | 0.07880 | 0.09023 | 1.5061 | 1.7246 | 1.0267 |
| 1.00 | 1.0 | 0.99773 | 1.01082 | 19.50408 | 19.50469 | 0.08710 | 0.10691 | 1.3624 | 1.6723 | 0.9871 |
| 0.67 | 1.5 | 1.49491 | 1.48728 | 19.48497 | 19.48863 | 0.15701 | 0.16794 | 1.6691 | 1.7854 | 1.0051 |
| 0.50 | 2.0 | 1.99628 | 1.98275 | 19.52412 | 19.52903 | 0.19447 | 0.26974 | 1.5508 | 2.1511 | 1.0068 |
| 0.40 | 2.5 | 2.51192 | 2.44339 | 19.44979 | 19.44432 | 0.22996 | 0.36457 | 1.4881 | 2.3592 | 1.0280 |
| 0.33 | 3.0 | 3.00626 | 2.91031 | 19.48570 | 19.48540 | 0.29471 | 0.52285 | 1.6011 | 2.8406 | 1.0330 |
| 0.25 | 4.0 | 4.02034 | 3.98695 | 19.52738 | 19.49352 | 0.35284 | 0.73595 | 1.3993 | 2.9186 | 1.0084 |
| 0.20 | 5.0 | 4.98604 | 4.80594 | 19.50534 | 19.54573 | 0.44138 | 1.08220 | 1.4521 | 3.5604 | 1.0375 |
| 0.13 | 7.5 | 7.51560 | 6.82938 | 19.44520 | 19.51234 | 0.68866 | 1.63264 | 1.5944 | 3.7799 | 1.1005 |
| 0.10 | 10.0 | 9.99919 | 8.82495 | 19.38640 | 19.44216 | 0.85783 | 2.27168 | 1.5369 | 4.0701 | 1.1331 |
| 0.05 | 20.0 | 19.97205 | 15.60862 | 19.57121 | 19.36086 | 1.50998 | 5.83760 | 1.5296 | 5.9134 | 1.2796 |
| 0.02 | 50.0 | 49.72038 | 28.43690 | 19.91299 | 20.26540 | 3.48672 | 15.16318 | 1.9387 | 8.4310 | 1.7484 |
| 0.01 | 100.0 | 100.48931 | 32.56446 | 15.76563 | 18.83286 | 4.57115 | 17.12406 | 1.3754 | 5.1523 | 3.0859 |

TABLE C-18

**MONTE CARLO RESULTS, SAMPLE SIZE 500, POPULATION MEAN 20**

| Relative Step $d/\gamma$ | GAMMA Pop. $\gamma$ | GAMMA Sample $g_s$ | GAMMA BCTM $g_B$ | MEAN Sample $m_s$ | MEAN BCTM $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10103 | — | 20.00105 | 19.99828 | 0.03642 | 0.00179 | 5.7580 | — | — |
| 6.67 | 0.15 | 0.15136 | — | 20.00299 | 20.00024 | 0.03226 | 0.00561 | 3.4002 | — | — |
| 4.00 | 0.25 | 0.24819 | — | 20.00185 | 20.00489 | 0.03339 | 0.01684 | 2.1116 | — | — |
| 3.00 | 0.333 | 0.33383 | — | 19.99719 | 20.00034 | 0.04419 | 0.02865 | 2.0980 | — | — |
| 2.00 | 0.5 | 0.50020 | 0.50423 | 19.99408 | 19.99114 | 0.06218 | 0.04401 | 1.9499 | 1.3600 | 0.9920 |
| 1.43 | 0.7 | 0.69952 | 0.70392 | 19.99949 | 19.98917 | 0.06998 | 0.06836 | 1.5720 | 1.5356 | 0.9938 |
| 1.18 | 0.85 | 0.84638 | 0.84950 | 19.98291 | 19.96656 | 0.08398 | 0.08290 | 1.5631 | 1.5429 | 0.9963 |
| 1.00 | 1.0 | 0.99988 | 0.97529 | 20.00776 | 20.01273 | 0.10623 | 0.11938 | 1.7222 | 1.9354 | 1.0252 |
| 0.67 | 1.5 | 1.49589 | 1.46969 | 19.98584 | 19.98600 | 0.15277 | 0.19780 | 1.6436 | 2.1280 | 1.0178 |
| 0.50 | 2.0 | 1.99017 | 1.89570 | 20.00676 | 20.00317 | 0.16854 | 0.28457 | 1.4057 | 2.3735 | 1.0498 |
| 0.40 | 2.5 | 2.50196 | 2.46917 | 19.98659 | 19.98877 | 0.23833 | 0.37851 | 1.5262 | 2.4238 | 1.0133 |
| 0.33 | 3.0 | 2.99170 | 2.88580 | 19.99297 | 19.96278 | 0.28960 | 0.52370 | 1.5867 | 2.8694 | 1.0367 |
| 0.25 | 4.0 | 4.00226 | 3.83451 | 20.03914 | 20.05341 | 0.36550 | 0.63228 | 1.5071 | 2.6072 | 1.0437 |
| 0.20 | 5.0 | 5.01119 | 4.66232 | 20.04012 | 19.98916 | 0.44355 | 1.07972 | 1.5042 | 3.6617 | 1.0748 |
| 0.13 | 7.5 | 7.50502 | 7.00770 | 20.00457 | 19.94182 | 0.61598 | 1.87484 | 1.3898 | 4.2302 | 1.0710 |
| 0.10 | 10.0 | 9.95838 | 8.64710 | 19.96651 | 19.95985 | 0.86366 | 2.29883 | 1.5792 | 4.2035 | 1.1516 |
| 0.05 | 20.0 | 19.91478 | 16.19875 | 19.93708 | 19.76068 | 1.91159 | 6.87492 | 1.8659 | 6.7105 | 1.2294 |
| 0.02 | 50.0 | 49.56723 | 29.85335 | 19.32074 | 19.94413 | 3.27419 | 12.74908 | 1.7318 | 6.7433 | 1.6586 |
| 0.01 | 100.0 | 100.26979 | 31.57581 | 20.40994 | 20.32106 | 5.07303 | 16.53015 | 2.2087 | 7.1970 | 3.1755 |

TABLE C-19

**MONTE CARLO RESULTS, SAMPLE SIZE 500, POPULATION MEAN 20.5**

| Relative Step d/γ | GAMMA | | | MEAN | | STD. DEV. | | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pop. γ | Sample $g_s$ | BCTN $g_B$ | Sample $m_s$ | BCTN $m_B$ | Mean $s_m$ | Gamma $s_g$ | | | |
| 10.00 | 0.1 | 0.10105 | 0.09998 | 20.49902 | 20.50080 | 0.01065 | 0.00523 | 1.6841 | 0.8270 | 1.0107 |
| 6.67 | 0.15 | 0.15046 | 0.14605 | 20.50142 | 20.50028 | 0.01790 | 0.01172 | 1.9377 | 1.2687 | 1.0302 |
| 4.00 | 0.25 | 0.24918 | 0.24906 | 20.49838 | 20.49600 | 0.02963 | 0.01987 | 1.8809 | 1.2613 | 1.0005 |
| 3.00 | 0.333 | 0.33362 | 0.33035 | 20.49473 | 20.48763 | 0.04794 | 0.02082 | 2.2943 | 0.9964 | 1.0099 |
| 2.00 | 0.5 | 0.50032 | 0.50468 | 20.50021 | 20.50143 | 0.05367 | 0.04753 | 1.6814 | 1.4890 | 0.9914 |
| 1.43 | 0.7 | 0.69538 | 0.70253 | 20.50497 | 20.49920 | 0.06520 | 0.07295 | 1.4675 | 1.6419 | 0.9898 |
| 1.18 | 0.85 | 0.84856 | 0.83216 | 20.49023 | 20.49469 | 0.08567 | 0.09327 | 1.6277 | 1.7721 | 1.0197 |
| 1.00 | 1.0 | 1.00057 | 0.98110 | 20.49628 | 20.49784 | 0.09081 | 0.11703 | 1.4635 | 1.8860 | 1.0198 |
| 0.67 | 1.5 | 1.50927 | 1.48571 | 20.48929 | 20.48860 | 0.14934 | 0.17423 | 1.5893 | 1.8543 | 1.0159 |
| 0.50 | 2.0 | 2.00750 | 1.94875 | 20.50893 | 20.49805 | 0.18737 | 0.26597 | 1.5202 | 2.1579 | 1.0302 |
| 0.40 | 2.5 | 2.49814 | 2.48819 | 20.48128 | 20.48706 | 0.25084 | 0.40482 | 1.5940 | 2.5724 | 1.0040 |
| 0.33 | 3.0 | 3.02268 | 2.90467 | 20.46101 | 20.46622 | 0.27383 | 0.45928 | 1.4906 | 2.5001 | 1.0406 |
| 0.25 | 4.0 | 4.00221 | 3.78673 | 20.52833 | 20.50912 | 0.40534 | 0.79401 | 1.6925 | 3.3153 | 1.0569 |
| 0.20 | 5.0 | 4.98820 | 4.65575 | 20.43031 | 20.40728 | 0.41983 | 0.86364 | 1.4258 | 2.9330 | 1.0714 |
| 0.13 | 7.5 | 7.52456 | 6.95626 | 20.46250 | 20.46332 | 0.66288 | 1.74320 | 1.5067 | 3.9623 | 1.0817 |
| 0.10 | 10.0 | 10.00594 | 9.26018 | 20.54495 | 20.48799 | 0.93575 | 2.39648 | 1.5978 | 4.0910 | 1.0805 |
| 0.05 | 20.0 | 19.97810 | 16.16709 | 20.29722 | 20.48159 | 2.04337 | 6.24297 | 1.9984 | 6.1056 | 1.2357 |
| 0.02 | 50.0 | 49.75421 | 30.97969 | 21.27439 | 20.68358 | 3.75028 | 14.20722 | 1.5691 | 5.9441 | 1.6060 |
| 0.01 | 100.0 | 100.12213 | 31.57729 | 21.25785 | 20.52370 | 4.51479 | 16.54175 | 2.1327 | 7.8140 | 3.1707 |

TABLE C-20

**MONTE CARLO RESULTS, SAMPLE SIZE 500, POPULATION MEAN 25**

| Relative Step $d/\gamma$ | GAMMA Pop. $\gamma$ | GAMMA Sample $g_s$ | GAMMA BCTN $g_B$ | MEAN Sample $m_s$ | MEAN BCTN $m_B$ | STD. DEV. Mean $s_m$ | STD. DEV. Gamma $s_g$ | G | H | BIAS $g_s/g_B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.00 | 0.1 | 0.10103 | — | 24.99921 | 24.99395 | 0.03316 | — | 5.2426 | - | — |
| 6.67 | 0.15 | 0.15121 | — | 25.00093 | 25.00075 | 0.03316 | — | 3.4951 | - | — |
| 4.00 | 0.25 | 0.25086 | — | 25.00010 | 25.00151 | 0.03866 | — | 2.4449 | - | — |
| 3.00 | 0.33 | 0.33407 | — | 24.99991 | 24.99713 | 0.04513 | — | 2.1405 | - | — |
| 2.00 | 0.5 | 0.50135 | 0.49733 | 24.99154 | 24.99066 | 0.05804 | 0.04990 | 1.4852 | 1.4388 | 1.0081 |
| 1.43 | 0.7 | 0.69944 | 0.70788 | 25.00396 | 25.00018 | 0.07452 | 0.07917 | 1.6645 | 1.7682 | 0.9881 |
| 1.18 | 0.85 | 0.85337 | 0.84416 | 24.99775 | 24.98990 | 0.09288 | 0.09456 | 1.7396 | 1.7[illegible] | 1.0109 |
| 1.00 | 1.0 | 0.99932 | 0.98905 | 25.00092 | 24.98520 | 0.08653 | 0.10611 | 1.3833 | 1.0786 | 1.0104 |
| 0.67 | 1.5 | 1.48564 | 1.50318 | 24.99525 | 24.97537 | 0.13951 | 0.19014 | 1.4674 | 2.0000 | 0.9883 |
| 0.50 | 2.0 | 1.98487 | 1.95397 | 25.01917 | 24.96882 | 0.18704 | 0.25677 | 1.5135 | 2.0778 | 1.0158 |
| 0.40 | 2.5 | 2.49541 | 2.47206 | 25.00345 | 24.95183 | 0.18762 | 0.36067 | 1.2000 | 2.3069 | 1.0094 |
| 0.33 | 3.0 | 3.00944 | 2.92395 | 24.97811 | 24.91607 | 0.25929 | 0.45265 | 1.4021 | 2.4477 | 1.0292 |
| 0.25 | 4.0 | 3.99531 | 3.93427 | 24.98334 | 24.95911 | 0.38191 | 0.66808 | 1.5349 | 2.6849 | 1.0155 |
| 0.20 | 5.0 | 4.99920 | 4.86478 | 24.98001 | 24.91989 | 0.47979 | 1.10358 | 1.5594 | 3.5868 | 1.0276 |
| 0.13 | 7.5 | 7.[illegible]688 | 7.43223 | 24.90103 | 24.83667 | 0.70551 | 1.92214 | 1.5009 | 4.0892 | 1.0076 |
| 0.10 | 10.0 | 10.06119 | 8.98490 | 24.94598 | 24.74085 | 0.91162 | 2.36253 | 1.6042 | 4.1575 | 1.1198 |
| 0.05 | 20.0 | 19.98140 | 17.10676 | 25.12019 | 24.73116 | 1.74682 | 6.34130 | 1.6145 | 5.8611 | 1.1680 |
| 0.02 | 50.0 | 50.39967 | 26.65554 | 26.01556 | 23.74781 | 3.52041 | 12.77265 | 2.0882 | 7.5764 | 1.8908 |
| 0.01 | 100.0 | 99.63984 | 33.12160 | 25.46837 | 22.16963 | 4.62088 | 17.12281 | 2.2059 | 8.1740 | 3.0083 |